# Operator Preconditioning: Theory, Practice, and Robustness

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**The Pennsylvania State University**

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences

Joint with: X. Hu and J. Adler (Tufts U),
F. Gaspar and C. Rodrigo (U Zaragoza),
J. Xu (Penn State), V. Petkov (U Bordeaux)

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- Further applications (not in this talk): Modelling of social and bio phenomena and networks (machine learning).

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Convergence rate of MINRES: $\|\mathbf{x} - \mathbf{x}^m\| \leq \delta \|\mathbf{x} - \mathbf{x}^{m-1}\|$

$$\delta = \frac{\kappa(\mathbf{A}) - 1}{\kappa(\mathbf{A}) + 1} \to 1 \text{ as the condition number } \kappa(\mathbf{A}) := \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \to \infty$$

# Algebraic systems arising from PDE discretizations

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- A "good" method should use the knowledge of some PDE properties:
    - Symmetry, smoothnes of the solution, types of singularities.
    - Geometric properties of the physical domains.
    - Special properties of the discretization methods.

# Numerical models and available fast solvers

**PDE model $\Longrightarrow$ numerical models:**

- Hilbert space $\mathcal{H}$ equipped with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$
- Operator $\mathbf{A} : \mathcal{H} \mapsto \mathcal{H}'$ and PDE: $Au = f$.
- Example: $A = -\Delta : H_0^1 \mapsto H^{-1}$ (Laplacian) on a bounded smooth domain with appropriate boundary conditions.
- Numerical model: restricting these equations to finite dimensional space $\mathcal{V}$, $\mathcal{V}'$ which "approximate" $\mathcal{H}$ and $\mathcal{H}'$.

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## Fast solvers (currently available):

- Advances in the recent years for $A = d^* K d$, where $d = \{\text{grad or curl or div}\,.$
- AMG for $d = \text{grad}$: Brandt, McCormick, Ruge 1983 (original); Vassilevski 2008 (monograph); Xu & Z. (Acta Numerica, May 2017).
- HX (Hiptmair & Xu 2007) preconditioners for $d = \text{curl}$, or $d = \text{div}$.

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Well-posedness (is $\mathbf{A}$ an isomorphism?):

Continuity of $\mathbf{A}$: $$\sup_{\mathbf{0}\neq\mathbf{x}\in\mathcal{H}} \sup_{\mathbf{0}\neq\mathbf{y}\in\mathcal{H}} \frac{\langle \mathbf{Ax}, \mathbf{y}\rangle}{\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}}} \leq \beta$$

Continuity of $\mathbf{A}^{-1}$ (inf-sup condition): $$\inf_{\mathbf{0}\neq\mathbf{x}\in\mathcal{H}} \sup_{\mathbf{0}\neq\mathbf{y}\in\mathcal{H}} \frac{\langle \mathbf{Ax}, \mathbf{y}\rangle}{\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}}} \geq \gamma > 0$$

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Example: Stokes equation

$$\mathbf{Ax} = \mathbf{f} \Longrightarrow \begin{pmatrix} -\Delta & \mathrm{div}^* \\ \mathrm{div} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$$

where $\mathcal{H} = [H_0^1]^3 \times L^2$, and $\|\mathbf{x}\|_{\mathcal{H}}^2 := \|\nabla \boldsymbol{u}\|^2 + \|p\|^2$

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Requirements on **B**: $\kappa(\mathbf{BA}) = \|BA\| \|(BA)^{-1}\| = \mathcal{O}(1) \ll \kappa(\mathbf{A})$

$\mathbf{B} \approx \mathbf{A}^{-1}$ and the action of $\mathbf{B}$ is easy to compute

- Apply Krylov iterative methods to the preconditioned system
- An example: multigrid preconditioner for $A = -\Delta$:

## Comparison (the timing in blue is an extrapolation):

| $\mathbf{A} \in \mathbb{R}^{N \times N}$ | $N = 2^{20}$ $\approx 1.0 \times 10^6$ | $N = 2^{24}$ $\approx 1.6 \times 10^7$ | $N = 2^{28}$ $\approx 2.6 \times 10^8$ | $N = 2^{32}$ $\approx 4.3 \times 10^9$ |
|---|---|---|---|---|
| TaihuLight | 6 s | 7 h | 3.3 yrs | 13690 yrs |
| Laptop (MG) | 0.3 s | 4.6 s | 63.5 s | 1015 s |

# Operator Preconditioning (Loghin & Wathen 2004, Mardal & Winther 2011)

Problem: Given $\mathbf{f} \in \mathcal{H}'$, find $\mathbf{u} \in \mathcal{H}$ such that $\mathbf{Au} = \mathbf{f}$. Here, $\mathbf{A} : \mathcal{H} \mapsto \mathcal{H}'$ is an isomorphism.

Riesz operator: $\mathbf{B} : \mathcal{H}' \mapsto \mathcal{H}$, such that for every $\mathbf{f} \in \mathcal{H}'$,

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Estimate $\kappa(\mathbf{BA})$:

$$\|\mathbf{BA}\| = \sup_{\mathbf{x}\in\mathcal{H}} \sup_{\mathbf{y}\in\mathcal{H}} \frac{|(\mathbf{BA}\mathbf{x}, \mathbf{y})_{\mathcal{H}}|}{\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}}} = \sup_{\mathbf{x}\in\mathcal{H}} \sup_{\mathbf{y}\in\mathcal{H}} \frac{|\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle|}{\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}}} \leq \beta$$

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Riesz operator is a robust preconditioner!

# This talk: operator preconditioners for Krylov methods

## The operator preconditioners presented here:

- Provably robust w.r.t different parameters.
- Motivated by the well-posedness of the discrete problems (Loghin & Wathen 2004, Mardal & Winther 2004, 2011, Zulehner 2011, Pestana & Wathen 2015)

## Our focus (two examples)

- PDEs modeling soil consolidation (Biot's model in poroelasticity)
- Asymptotically disappearing solutions (ADS) of Maxwell's equations (Petkov et al. 2011,...; Cakoni et al. 2016).

# Poroelasticity problem

- This problem was firstly studied by Terzaghi in 1925. (*Terzaghi, Erdbaumechanik and Bodenphysikalischen Grundlagen. Deuticke, Leipzig, 1925*).
- The general three-dimensional theory was given by Biot in 1941. (*Biot M., General theory of three dimensional consolidation.* J. Appl. Phys. **12** *(1941), 155-169.*)
- The term *poroelasticity* was firstly coined by J. Geertsma in 1966. (*Geertsma J., Problems of rocks mechanics in petroleum production engineering, 1966*).
- Biot's models are used to study problems in geomechanics, hydrogeology, petrol engineering, biomechanics, etc.

# First example: Poroelasticity

Two-field (displacement-pressure) formulation:

$$-\operatorname{div} \boldsymbol{\sigma} + \alpha \nabla p = \boldsymbol{f}, \quad \boldsymbol{\sigma} = 2\mu\varepsilon(\boldsymbol{u}) + \lambda \operatorname{div}(\boldsymbol{u}) I \qquad \text{(Linear Elasiticy)}$$

$$-\alpha \operatorname{div} \partial_t \boldsymbol{u} + \operatorname{div} K \nabla p = g \qquad \text{(Darcy's Law)}$$

$\boldsymbol{\sigma}$: effective stress; $\boldsymbol{\varepsilon}$: effective strain; $\lambda$ and $\mu$: Lamé coefficients;
$K$: hydraulic conductivity; $\alpha$: Biot-Willis constant.

Terzaghi 1925, Biot 1941, Geertsma 1966

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Three-field (displacement-velocity-pressure) formulation:

$$-\operatorname{div}\boldsymbol{\sigma} + \alpha\nabla p = \boldsymbol{f}, \quad \boldsymbol{\sigma} = 2\mu\varepsilon(\boldsymbol{u}) + \lambda\operatorname{div}(\boldsymbol{u})I \qquad \text{(Linear Elasiticy)}$$

$$-\alpha\operatorname{div}\partial_t\boldsymbol{u} - \operatorname{div}\boldsymbol{w} = g \qquad \text{(Continuity)}$$

$$\boldsymbol{w} + K\nabla p = 0 \qquad \text{(Darcy's Law)}$$

$\boldsymbol{\sigma}$: effective stress; $\boldsymbol{\varepsilon}$: effective strain; $\lambda$ and $\mu$: Lamé coefficients;
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# Example of Coupled Problem: Maxwell's system

Maxwell's system: Let $\mathcal{O}$ be a bounded and connected domain, we consider the Maxwell's equation in the exterior of $\bar{\mathcal{O}}$, i.e. $\Omega = \mathbb{R}^3 \backslash \bar{\mathcal{O}}$

$$
\begin{aligned}
\boldsymbol{B}_t + \operatorname{curl} \boldsymbol{E} &= 0 && \text{[Faraday's Law ]} \\
\varepsilon \boldsymbol{E}_t - \operatorname{curl} \mu^{-1} \boldsymbol{B} &= -\boldsymbol{j} && \text{[Ampere's Law]} \\
\operatorname{div} \varepsilon \boldsymbol{E} &= \boldsymbol{\rho} && \text{[Gauß's law]} \\
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$\varepsilon$ - permitivity of the medium; $\mu$ - magnetic permeability

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Remark: computationally, we use $\Omega = \mathcal{S} \backslash \bar{\mathcal{O}}$ where $\mathcal{S}$ is a ball in $\mathbb{R}^3$ with sufficiently large radius. We also impose impedance boundary condition on the boundary of $\mathcal{O}$:

$$(1+\gamma)\boldsymbol{E}_{\tan} = -\boldsymbol{n} \times \boldsymbol{B}_{\tan}$$

# Linear Systems in Block Form

After discretization: $\mathbf{A}\mathbf{u} = \mathbf{f}$

Poroelasiticy: $\mathbf{A} = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\,\text{grad} \\ -\alpha\,\text{div} & -\tau\,\text{div}\,K\,\text{grad} \end{pmatrix}$ or $\begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\,\text{grad} & 0 \\ -\alpha\,\text{div} & 0 & -\tau\,\text{div} \\ 0 & \tau\,\text{grad} & \tau\mathcal{I}_{\boldsymbol{w}} \end{pmatrix}$

Maxwell's: $\mathbf{A} = \begin{pmatrix} 2\mathcal{I}_{\boldsymbol{B}} & \tau\,\text{curl} & \\ -\tau\,\text{curl} & 2\mathcal{I}_{\boldsymbol{E}} + \tau\mathcal{Z} & \text{grad} \\ & \tau\,\text{div} & 2\mathcal{I}_p \end{pmatrix}$

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## Properties of the linear systems:

- Large, indefinite, and with strong coupling: Difficult to solve for either direct methods or traditional iterative methods
- Many physical parameters $(\lambda, \mu, K, \alpha, \mu, \varepsilon)$ and discretization parameters $(h,\ \tau)$: Challenging to design linear solvers that are robust with respect to the parameters

# An incomplete literature review

Discretizations and Preconditioners for Poroelasticity:

- Well posedness and convergence Murad, Loula, Thomée 94–96
- ILU (Gambolati, Pini, & Ferronato 2003)
- Domain decomposition methods (Girault, Pencheva, Wheeler, & Wildey 2011; Florez, Wheeler, &Rodriguez 2013)
- Multigrid methods (Gaspar, Lisbona, & Rodrigo 2009)
- Approximate block factorization (Vassilevski, Lazarov 1996; Vassilevski 2008, Janna, Ferronato, & Gambolati 2009, White, Castelletto, & Tchelepi 2016, Castelletto, White, & Ferronato 2016)
- Parameter-robust preconditioning ( Axelsson, Padiy 1999, Vassilevski 2008, Axelsson, Blaheta, Neytcheva 2016; Lee, Mardal, & Winther 2015; Baerland, Lee, Mardal& Winther 2017; Hong & Kraus 2017)

Preconditioners for the full Maxwell's system (used also in Magnetohydrodynamics (MHD) simulations):

- ILU (Tóth, Keppens, & Botchev 1998, 1999; Badia, Martin, & Planas 2014)
- Domain decomposition methods (Ovtchinikov, Dobrain, Cai, & Keyes 2007; Reynolds, Samtaney, & Tiedeman, 2012)
- Multigrid methods (Shadid, Pawlovski, Banks, Chacón, Lin, & Tuminaro 2010; Benson, Adler, Cyr, MacLachlan, & Tuminaro, 2015)
- Approximate block factorization (Shadid, Cyr, Pawlovski, Tuminaro, Chacón, & Lin, 2010; Cyr, Shadid, Tuminaro, Pawlowski, & Chacón 2013; Phillips, Elman, Cyr, Shadid, & Pawlowski 2014)

# Norm-equivalent Preconditioner

Preconditioner $\mathbf{M} : \mathcal{H}' \mapsto \mathcal{H}$ is symmetric positive definite ($\mathbf{MA} : \mathcal{H} \mapsto \mathcal{H}$)

Loghin & Wathen 2004, Mardal & Winther 2004, 2011, Pestana & Wathen 2015

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**Norm-equivalence**

If $\mathbf{A}$ is well-posed w.r.t the norm $\|\cdot\|_{\mathcal{H}}$. Choose $\mathbf{M}$ such that

$$c_1\|\mathbf{x}\|_{\mathcal{H}}^2 \leq \|\mathbf{x}\|_{\mathbf{M}^{-1}}^2 \leq c_2\|\mathbf{x}\|_{\mathcal{H}}^2$$

then $\mathbf{M}$ and $\mathbf{A}$ are norm-equivalent and $\kappa(\mathbf{MA}) \leq \frac{c_2\beta}{c_1\gamma}$

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## Theorem (Preconditioned MINRES)

*The following estimate holds:*

$$\|\boldsymbol{A}(\mathbf{x}-\mathbf{x}^m)\|_{\boldsymbol{M}} \leq 2\delta^m\|\boldsymbol{A}(\mathbf{x}-\mathbf{x}^0)\|_{\boldsymbol{M}}, \quad \delta = \frac{\kappa(\mathbf{MA})-1}{\kappa(\mathbf{MA})+1}$$

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Question: How do we choose $\mathbf{M}$?

Loghin & Wathen 2004, Mardal & Winther 2004, 2011, Pestana & Wathen 2015

# Example: Stokes Equation

Well-posedness: well-posed w.r.t the norm $\|\mathbf{x}\|_{\mathcal{H}}^2 = \|\nabla \boldsymbol{u}\|^2 + \|p\|^2$

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# Field-of-Values Equivalent Preconditioner

Preconditioner $\mathbf{M}_L : \mathcal{H}' \mapsto \mathcal{H}$ is a general operator ($\mathbf{M}_L\mathbf{A} : \mathcal{H} \mapsto \mathcal{H}$)

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## Field-of-Values equivalence

The operators $\mathbf{M}_L$ and $\mathbf{A}$ are FoV-equivalent, if for any $\mathbf{x} \in \mathcal{H}$,

$$\sigma \leq \frac{(\mathbf{M}_L\mathbf{A}\mathbf{x}, \mathbf{x})_{\mathcal{H}}}{(\mathbf{x}, \mathbf{x})_{\mathcal{H}}}, \quad \frac{\|\mathbf{M}_L\mathbf{A}\mathbf{x}\|_{\mathcal{H}}}{\|\mathbf{x}\|_{\mathcal{H}}} \leq \Upsilon.$$

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## Theorem (Convergence of preconditioned GMRES)

*If $\mathbf{x}^m$ is the m-th iteration of GMRES method and $\mathbf{x}$ is the exact solution,*

$$\frac{\|\mathbf{M}_L\mathbf{A}(\mathbf{x} - \mathbf{x}^m)\|_{\mathcal{H}}}{\|\mathbf{M}_L\mathbf{A}(\mathbf{x} - \mathbf{x}^0)\|_{\mathcal{H}}} \leq \left(1 - \frac{\sigma^2}{\Upsilon^2}\right)^{m/2}$$

Loghin & Wathen 2004

# Field-of-Values Equivalent Preconditioner

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Question: How do we choose $\mathbf{M}_L$?

Loghin & Wathen 2004

# Example: Stokes Equation

Well-posedness: well-posed w.r.t the norm $\|\mathbf{x}\|_{\mathcal{H}}^2 = \|\nabla \boldsymbol{u}\|^2 + \|\boldsymbol{p}\|^2$

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- Inexact lower triangular preconditioner: “cheap” but “good” blocks

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- Inexact lower triangular preconditioner: "cheap" but "good" blocks

$$\mathbf{M}_L = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}}^{-1} & 0 \\ \mathrm{div} & \rho \mathcal{Q}_p^{-1} \end{pmatrix}^{-1}$$

  - Theory requires a parameter $\rho > \rho_0$ (Loghin & Wathen 2004)
  - In practice, $\rho = 1$

# Remove $\rho$

Idea: use $\|\cdot\|_{\mathbf{M}^{-1}}$ induced by $\mathbf{M} = \mathrm{diag}(\mathcal{Q}_{\boldsymbol{u}}, \mathcal{Q}_p)$ instead of $\|\cdot\|_{\mathcal{H}}$

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$$\sigma \leq \frac{(\mathbf{x}, \mathbf{M}_L \mathbf{A}\mathbf{x})_{\mathbf{M}^{-1}}}{(\mathbf{x}, \mathbf{x})_{\mathbf{M}^{-1}}}, \quad \frac{\|\mathbf{M}_L \mathbf{A}\mathbf{x}\|_{\mathbf{M}^{-1}}}{\|\mathbf{x}\|_{\mathbf{M}^{-1}}} \leq \Upsilon.$$

*provided* $\|\mathcal{I}_{\boldsymbol{u}} - \mathcal{Q}_{\boldsymbol{u}}\mathcal{A}\|_{\mathcal{A}} \leq \delta < 1$.

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- $\mathcal{Q}_{\boldsymbol{u}}$ needs to be convergent
- $\sigma$ and $\Upsilon$ only depend on $\beta$, $\gamma$, $\delta$, and the quality of spectral equivalent approximations
- Preconditioned GMRES converges uniformly

# Right Preconditioning

Preconditioner $\mathbf{M}_U : \mathcal{H}' \mapsto \mathcal{H}$ is a general operator ($\mathbf{A}\mathbf{M}_U : \mathcal{H}' \mapsto \mathcal{H}'$)

$$\mathbf{A}\mathbf{x} = \mathbf{f} \Longrightarrow \mathbf{A}\mathbf{M}_U\mathbf{x}' = \mathbf{f}, \quad \mathbf{x}' = \mathbf{M}_U^{-1}\mathbf{x} \in \mathcal{H}'$$

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Example: Stokes equation

- Upper triangular preconditioner: Riesz operator + upper triangular

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- Inexact upper triangular preconditioner:

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# Poroelasticity: Numerical Difficulties

## Terzaghi's problem

$$-E\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0,\ x \in (0,1),$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - K\frac{\partial^2 p}{\partial x^2} = 0.$$

$$E\frac{\partial u}{\partial x} = -1,\ p = 0,\ \text{on } x = 0,$$

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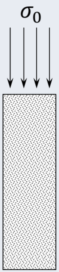

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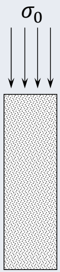
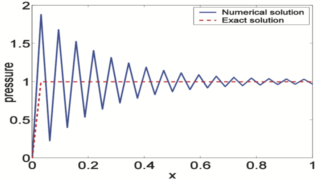



P1-P1 + Implicit Euler



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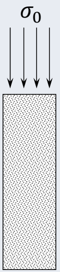


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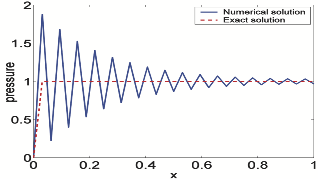
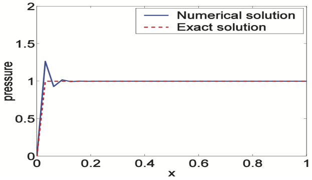



P2-P1 + Implicit Euler



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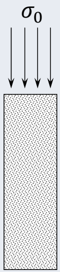


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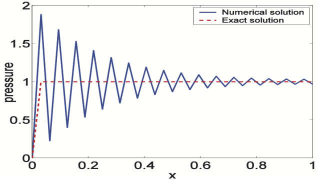


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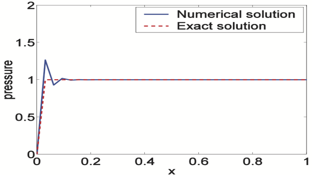


- Pressure may have nonphysical oscillations (low permeability/small time step)
- Using Stokes-stable pair reduces the oscillations but does not solve the problem

# Two-field Formulation with Stabilization

Stabilized scheme: Find $\boldsymbol{u}_h^n \in \boldsymbol{V}_h \subset \boldsymbol{H}_D^1$ and $p_h^n \in Q_h \subset H_{D,p}^1$

$$a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) - \alpha(\operatorname{div} \boldsymbol{v}_h, p_h^n) = (f(t_n), \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h$$

$$-\alpha(\operatorname{div} \overline{\partial}_t \boldsymbol{u}_h^n, q_h) - a_p(p_h^n, q_h) - \underbrace{\epsilon h^2(\nabla \overline{\partial}_t p_h^n, \nabla q_h)}_{\text{stabilization term}} = 0, \quad \forall q_h \in Q_h$$

where $\overline{\partial}_t \boldsymbol{u}_h^n := (\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})/\tau$, $\overline{\partial}_t p_h^n := (p_h^n - p_h^{n-1})/\tau$, and

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Brezzi & Pitkäranta 1984; Aguilar, Gaspar, Lisbona, & Rodrigo 2008; Rodrigo, Gaspar, Hu, & Z. 2016

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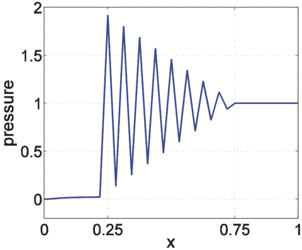
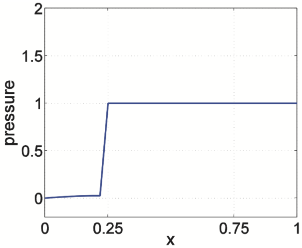
- Stokes-stable: MINI element/P2-P1
- Stokes-unstable: P1-P1(+ stabilization)

Brezzi & Pitkäranta 1984; Aguilar, Gaspar, Lisbona, & Rodrigo 2008; Rodrigo, Gaspar, Hu, & Z. 2016

# Effectiveness of the Stabilization




P1-P1 without stabilization

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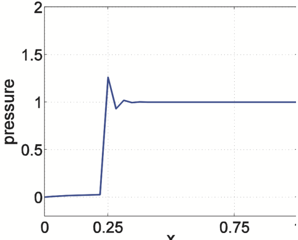
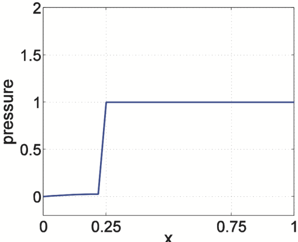



P2-P1 without stabilization




P2-P1 with stabilization

# Well-posedness

Linear system: $\mathbf{A}\mathbf{x} = \mathbf{f}$, $\boldsymbol{x} = (\boldsymbol{u}, p)^T$

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\mathcal{B}^T \\ \alpha\mathcal{B} & -\tau\mathcal{A}_p - \varepsilon h^2 \mathcal{L}_p \end{pmatrix}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) \to \mathcal{A}_{\boldsymbol{u}}$, $-(\operatorname{div} \boldsymbol{u}, p) \to \mathcal{B}$, $(K\nabla p, \nabla q) \to \mathcal{A}_p$, $(\nabla p, \nabla q) \to \mathcal{L}_p$

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A standard approach: assume Stokes-stable pair and $\varepsilon = 0$

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- Well-posedness follows from the fact that $\mathcal{A}_{\boldsymbol{u}}$ and $\mathcal{A}_p$ are SPD

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Linear system: $\mathbf{A}\mathbf{x} = \mathbf{f}$, $\boldsymbol{x} = (\boldsymbol{u}, p)^T$

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- Well-posedness follows from the fact that $\mathcal{A}_{\boldsymbol{u}}$ and $\mathcal{A}_p$ are SPD
- Argument fails as $\tau \to 0 \not\Longrightarrow$ robust block preconditioners
  For example, it is well-known that the preconditioner
  $$\begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\mathcal{B}^T \\ 0 & \tau\mathcal{A}_p \end{pmatrix}$$
  only works when coupling is weak and diverges for many cases of practical interest (White, Castelletto, & Tchelepi 2016)

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$$\|\boldsymbol{u}\|_{\mathcal{A}_{\boldsymbol{u}}}^2 := 2\mu\|\varepsilon(\boldsymbol{u})\|^2 + \lambda\|\operatorname{div}\boldsymbol{u}\|^2, \ \|p\|_{\mathcal{A}_p}^2 := (K\nabla p, \nabla p), \ \|p\|_{\mathcal{L}_p}^2 := \|\nabla p\|^2$$

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**Theorem** (Adler, Gaspar, Hu, Rodrigo, & Z. 2017)

*Assume $\varepsilon = \frac{\theta\alpha^2}{\lambda+2\mu/d}$ with $\theta > 0$, then the linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$. The constants $\gamma$ and $\beta$ do not depend on the physical parameters ($\lambda$, $\mu$, $\alpha$, $K$) and the discretization parameters ($h$, $\tau$).*

# Norm-equivalent Preconditioner

Weighted norm: $\|\boldsymbol{x}\|_{\mathcal{H}}^2 = \|\boldsymbol{u}\|_{\mathcal{A}_{\boldsymbol{u}}}^2 + \tau\|\boldsymbol{p}\|_{\mathcal{A}_p}^2 + \varepsilon \boldsymbol{h}^2\|\boldsymbol{p}\|_{\mathcal{L}_p}^2 + \frac{\alpha^2}{\lambda+2\mu/d}\|\boldsymbol{p}\|^2$

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Norm-equivalent preconditioner: block diagonal

$$\mathbf{B}_D = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & 0 \\ 0 & \tau\mathcal{A}_p + \varepsilon h^2\mathcal{L}_p + \frac{\alpha^2}{\lambda+2\mu/d}\mathcal{I}_p \end{pmatrix}^{-1} \text{ and } \mathbf{M}_D = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}} & 0 \\ 0 & \mathcal{Q}_p \end{pmatrix}$$

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**Theorem** (Adler, Gaspar, Hu, Rodrigo, & Z. 2017)

*If the linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$, then we have*

$$\kappa(\mathbf{B}_D\mathbf{A}) \leq C_1 \text{ and } \kappa(\mathbf{M}_D\mathbf{A}) \leq C_2$$

*where $C_1$ and $C_2$ are constants independent of the physical parameters ($\lambda$, $\mu$, $\alpha$, $K$) and the discretization parameters ($h$, $\tau$).*

# FoV-equivalent Preconditioners: Left Preconditioning

Block lower triangular preconditioners:

$$\mathbf{B}_L = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & 0 \\ -\alpha\mathcal{B} & \tau\mathcal{A}_p + \varepsilon h^2\mathcal{L}_p + \frac{\alpha^2}{\lambda+2\mu/d}\mathcal{I}_p \end{pmatrix}^{-1} \text{ and } \mathbf{M}_L = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}}^{-1} & 0 \\ -\alpha\mathcal{B} & \mathcal{Q}_p^{-1} \end{pmatrix}^{-1}$$

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**Theorem** (Adler, Gaspar, Hu, Rodrigo, & Z. 2017)

*If the linear system is well-posed w.r.t* $\|\cdot\|_{\mathcal{H}}$, *then we have*

$$\sigma \le \frac{(\mathbf{B}_L\mathbf{A}\boldsymbol{x},\boldsymbol{x})_{\mathbf{B}_D^{-1}}}{(\boldsymbol{x},\boldsymbol{x})_{\mathbf{B}_D^{-1}}}, \quad \frac{\|\mathbf{B}_L\mathbf{A}\boldsymbol{x}\|_{\mathbf{B}_D^{-1}}}{\|\boldsymbol{x}\|_{\mathbf{B}_D^{-1}}} \le \Upsilon$$

*and, if* $\|I - \mathcal{Q}_{\boldsymbol{u}}\mathcal{A}_{\boldsymbol{u}}\|_{\mathcal{A}_{\boldsymbol{u}}} \le \delta < 1$, *we have*

$$\sigma \le \frac{(\mathbf{M}_L\mathbf{A}\boldsymbol{x},\boldsymbol{x})_{\mathbf{M}_D^{-1}}}{(\boldsymbol{x},\boldsymbol{x})_{\mathbf{M}_D^{-1}}}, \quad \frac{\|\mathbf{M}_L\mathbf{A}\boldsymbol{x}\|_{\mathbf{M}_D^{-1}}}{\|\boldsymbol{x}\|_{\mathbf{M}_D^{-1}}} \le \Upsilon$$

*where* $\sigma$ *and* $\Upsilon$ *are constants independent of the physical parameters* $(\lambda,\ \mu,\ \alpha,\ K)$ *and the discretization parameters* $(h,\ \tau)$.

# FoV-equivalent Preconditioners: Right Preconditioning

Block upper triangular preconditioners:

$$\mathbf{B}_U = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\mathcal{B}^T \\ 0 & \tau\mathcal{A}_p + \varepsilon \boldsymbol{h}^2\mathcal{L}_p + \frac{\alpha^2}{\lambda+2\mu/d}\mathcal{I}_p \end{pmatrix}^{-1} \text{ and } \mathbf{M}_U = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}}^{-1} & \alpha\mathcal{B}^T \\ 0 & \mathcal{Q}_p^{-1} \end{pmatrix}^{-1}$$

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*If the linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$, then we have*

$$\sigma \le \frac{(\mathbf{A}\mathbf{B}_U \boldsymbol{x}', \boldsymbol{x}')_{\mathbf{B}_D}}{(\boldsymbol{x}', \boldsymbol{x}')_{\mathbf{B}_D}}, \quad \frac{\|\mathbf{A}\mathbf{B}_U \boldsymbol{x}'\|_{\mathbf{B}_D}}{\|\boldsymbol{x}'\|_{\mathbf{B}_D}} \le \Upsilon$$

*and, if $\|\mathcal{I}_{\boldsymbol{u}} - \mathcal{Q}_{\boldsymbol{u}} \mathcal{A}_{\boldsymbol{u}}\|_{\mathcal{A}_{\boldsymbol{u}}} \le \delta < 1$, we have*

$$\sigma \le \frac{(\mathbf{A}\mathbf{M}_U \boldsymbol{x}', \boldsymbol{x}')_{\mathbf{M}_D}}{(\boldsymbol{x}', \boldsymbol{x}')_{\mathbf{M}_D}}, \quad \frac{\|\mathbf{A}\mathbf{M}_U \boldsymbol{x}'\|_{\mathbf{M}_D}}{\|\boldsymbol{x}'\|_{\mathbf{M}_D}} \le \Upsilon$$

*where $\sigma$ and $\Upsilon$ are constants independent of the physical parameters $(\lambda, \mu, \alpha, K)$ and the discretization parameters $(h, \tau)$.*

# Relationship with a Sequential-implicit Scheme

Fixed-stress splitting scheme: assume Stokes-stable pair and $\varepsilon = 0$

$$\begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\mathcal{B} \\ 0 & \mathcal{S}_p \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\alpha\mathcal{B} & \frac{\alpha^2}{\tilde{K}}\mathcal{I}_p \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^{k} \\ p^{k} \end{pmatrix} + \begin{pmatrix} f_{\boldsymbol{u}} \\ f_p \end{pmatrix}$$

where $\mathcal{S}_p = \tau\mathcal{A}_p + \frac{\alpha^2}{\tilde{K}}\mathcal{I}_p$, $\tilde{K}$ is a suitable bulk modulus

Kim, Tchelepi, & Juanes 2011; White, Castelletto, & Tchelepi 2015

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Recast as linear iterative method:

$$\begin{pmatrix} \boldsymbol{u}^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^{k} \\ p^{k} \end{pmatrix} + \underbrace{\begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\mathcal{B} \\ 0 & \mathcal{S}_p \end{pmatrix}^{-1}}_{\mathbf{B}_U} \left[ \begin{pmatrix} f_{\boldsymbol{u}} \\ f_p \end{pmatrix} - \mathbf{A} \begin{pmatrix} \boldsymbol{u}^{k} \\ p^{k} \end{pmatrix} \right]$$

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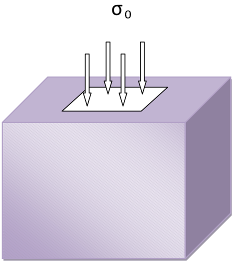
Our analysis provides:

- Theory of using fixed-stress splitting scheme as a preconditioner
- A good choice of $\tilde{K}$: $\tilde{K} = \lambda + 2\mu/d$
- Inexact version of fixed-stress splitting scheme and its analysis

Kim, Tchelepi, & Juanes 2011; White, Castelletto, & Tchelepi 2015

# 3D Footing Probelm




- Uniform porous material
- Apply uniform load on the top
- Drained on all sides
- Tolerance of Krylov iterative methods is $10^{-6}$

# Performance of Preconditioner: $K = 10^{-6}$ and $\nu = 0.2$

(1) Preconditioner $\mathbf{B}_D$

| $\tau$ \ $h$ | 1/4 | 1/8 | 1/16 | 1/32 |
|---|---|---|---|---|
| 0.1 | 7 | 7 | 8 | * |
| 0.01 | 7 | 7 | 8 | * |
| 0.001 | 7 | 7 | 8 | * |
| 0.0001 | 7 | 7 | 8 | * |

(2) Preconditioner $\mathbf{B}_L$

| $\tau$ \ $h$ | 1/4 | 1/8 | 1/16 | 1/32 |
|---|---|---|---|---|
| 0.1 | 5 | 5 | 6 | * |
| 0.01 | 5 | 5 | 6 | * |
| 0.001 | 5 | 5 | 6 | * |
| 0.0001 | 5 | 5 | 6 | * |

(3) Preconditioner $\mathbf{B}_U$

| $\tau$ \ $h$ | 1/4 | 1/8 | 1/16 | 1/32 |
|---|---|---|---|---|
| 0.1 | 4 | 4 | 4 | * |
| 0.01 | 4 | 4 | 5 | * |
| 0.001 | 5 | 5 | 6 | * |
| 0.0001 | 5 | 5 | 6 | * |

(4) Preconditioner $\mathbf{M}_D$

| $\tau$ \ $h$ | 1/4 | 1/8 | 1/16 | 1/32 |
|---|---|---|---|---|
| 0.1 | 8 | 8 | 9 | 9 |
| 0.01 | 8 | 8 | 9 | 9 |
| 0.001 | 8 | 8 | 9 | 9 |
| 0.0001 | 8 | 8 | 9 | 9 |

(5) Prconditioner $\mathbf{M}_L$

| $\tau$ \ $h$ | 1/4 | 1/8 | 1/16 | 1/32 |
|---|---|---|---|---|
| 0.1 | 6 | 6 | 8 | 8 |
| 0.01 | 6 | 6 | 8 | 8 |
| 0.001 | 6 | 6 | 8 | 8 |
| 0.0001 | 7 | 6 | 8 | 8 |

(6) Preconditioner $\mathbf{M}_U$

| $\tau$ \ $h$ | 1/4 | 1/8 | 1/16 | 1/32 |
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| 0.0001 | 6 | 7 | 8 | 8 |

- The symbol $*$ in the tables on the left is for cases when the direct solver ran out of memory.

# Performance of Preconditioners: Varying $K$ and $\nu$

Table: Varying $K$: $\nu = 0.2$, $h = 1/16$ and $\tau = 0.01$

| | 1 | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ | $10^{-8}$ | $10^{-10}$ |
|---|---|---|---|---|---|---|
| $\mathbf{B}_D$ | 4 | 7 | 8 | 8 | 8 | 8 |
| $\mathbf{B}_L$ | 2 | 5 | 6 | 6 | 6 | 6 |
| $\mathbf{B}_U$ | 3 | 4 | 5 | 5 | 5 | 5 |
| $\mathbf{M}_D$ | 5 | 8 | 9 | 9 | 9 | 9 |
| $\mathbf{M}_L$ | 5 | 7 | 8 | 8 | 8 | 8 |
| $\mathbf{M}_U$ | 5 | 7 | 8 | 8 | 9 | 8 |

Table: Varying $\nu$: $K = 10^{-6}$, $h = 1/16$ and $\tau = 0.01$

| | 0.1 | 0.2 | 0.4 | 0.45 | 0.49 | 0.499 |
|---|---|---|---|---|---|---|
| $\mathbf{B}_D$ | 7 | 8 | 11 | 11 | 12 | 12 |
| $\mathbf{B}_L$ | 5 | 6 | 8 | 8 | 8 | 9 |
| $\mathbf{B}_U$ | 4 | 5 | 6 | 6 | 5 | 4 |
| $\mathbf{M}_D$ | 8 | 9 | 12 | 13 | 14 | 13 |
| $\mathbf{M}_L$ | 7 | 8 | 11 | 11 | 12 | 12 |
| $\mathbf{M}_U$ | 7 | 8 | 7 | 8 | 17 | 11 |

# Three-field Formulation

Choices of the finite-element spaces:

- Displacement $\boldsymbol{u}$: Crouzeix-Raviart element
- Darcy velocity $\boldsymbol{w}$: lowest order Raviart-Thomas element
- Pressure $p$: piecewise constant element

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$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha\mathcal{B}_{\boldsymbol{u}}^T & 0 \\ -\alpha\mathcal{B}_{\boldsymbol{u}} & 0 & -\tau\mathcal{B}_{\boldsymbol{w}} \\ 0 & \tau\mathcal{B}_{\boldsymbol{w}}^T & \tau\mathcal{I}_{\boldsymbol{w}} \end{pmatrix}$$

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Weighted norm:

$$\|\boldsymbol{x}\|_{\mathcal{H}}^2 := \|\boldsymbol{u}\|_{\mathcal{A}_{\boldsymbol{u}}}^2 + \alpha^2 \left(\lambda + \frac{2\mu}{d}\right)^{-1} \|p\|^2 + \tau\|\boldsymbol{w}\|_{K^{-1}}^2 + \frac{\tau^2}{\alpha^2}\left(\lambda + \frac{2\mu}{d}\right) \|\nabla \cdot \boldsymbol{w}\|^2$$

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# Block Preconditioners

Norm-equivalent preconditioners: block diagonal

$$\mathbf{B}_D = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & 0 & 0 \\ 0 & \frac{\alpha^2}{\lambda + \frac{2\mu}{d}} \mathcal{I}_p & 0 \\ 0 & 0 & \tau \mathcal{I}_{\boldsymbol{w}} + \frac{\tau^2}{\alpha^2} \left(\lambda + \frac{2\mu}{d}\right) \mathcal{A}_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

FoV-equivalent preconditioners: block triangular

$$\mathbf{B}_L = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & 0 & 0 \\ -\alpha \mathcal{B}_{\boldsymbol{u}} & \frac{\alpha^2}{\lambda + \frac{2\mu}{d}} \mathcal{I}_p & 0 \\ 0 & \tau \mathcal{B}_{\boldsymbol{w}}^T & \tau \mathcal{I}_{\boldsymbol{w}} + \frac{\tau^2}{\alpha^2} \left(\lambda + \frac{2\mu}{d}\right) \mathcal{A}_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

$$\mathbf{B}_U = \begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha \mathcal{B}_{\boldsymbol{u}}^T & 0 \\ 0 & \frac{\alpha^2}{\lambda + \frac{2\mu}{d}} \mathcal{I}_p & -\tau \mathcal{B}_{\boldsymbol{w}} \\ 0 & 0 & \tau \mathcal{I}_{\boldsymbol{w}} + \frac{\tau^2}{\alpha^2} \left(\lambda + \frac{2\mu}{d}\right) \mathcal{A}_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

# Block Preconditioners

Norm-equivalent preconditioners: block diagonal

$$\mathbf{M}_D = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}} & 0 & 0 \\ 0 & \mathcal{Q}_p & 0 \\ 0 & 0 & \mathcal{Q}_{\boldsymbol{w}} \end{pmatrix}$$

FoV-equivalent preconditioners: block triangular

$$\mathbf{M}_L = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}}^{-1} & 0 & 0 \\ -\alpha \mathcal{B}_{\boldsymbol{u}} & \mathcal{Q}_p^{-1} & 0 \\ 0 & \tau \mathcal{B}_{\boldsymbol{w}}^T & \mathcal{Q}_{\boldsymbol{w}}^{-1} \end{pmatrix}^{-1}$$

$$\mathbf{M}_U = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{u}}^{-1} & \alpha \mathcal{B}_{\boldsymbol{u}}^T & 0 \\ 0 & \mathcal{Q}_p^{-1} & -\tau \mathcal{B}_{\boldsymbol{w}} \\ 0 & 0 & \mathcal{Q}_{\boldsymbol{w}}^{-1} \end{pmatrix}^{-1}$$

$\mathcal{Q}_{\boldsymbol{u}}$: DD/MG $\quad$ $\mathcal{Q}_p$: Jacobi $\quad$ $\mathcal{Q}_{\boldsymbol{w}}$: Hiptmair-Xu

# Fixed-Stress Splitting Scheme for Three-field Formulation

Linear iterative method:

$$\begin{pmatrix} \boldsymbol{u}^{k+1} \\ p^{k+1} \\ \boldsymbol{w}^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^{k} \\ p^{k} \\ \boldsymbol{w}^{k} \end{pmatrix} + \mathbf{B}_U \left[ \begin{pmatrix} f_{\boldsymbol{u}} \\ f_p \\ f_{\boldsymbol{w}} \end{pmatrix} - \mathbf{A} \begin{pmatrix} \boldsymbol{u}^{k} \\ p^{k} \\ \boldsymbol{w}^{k} \end{pmatrix} \right]$$

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"Fixed-stress" splitting scheme:

$$\begin{pmatrix} \mathcal{A}_{\boldsymbol{u}} & \alpha \mathcal{B}_{\boldsymbol{u}}^T & 0 \\ 0 & \frac{\alpha^2}{\lambda + \frac{2\mu}{d}} \mathcal{I}_p & -\tau \mathcal{B}_{\boldsymbol{w}} \\ 0 & 0 & \tau \mathcal{I}_{\boldsymbol{w}} + \tau^2 \frac{\lambda + \frac{2\mu}{d}}{\alpha^2} \mathcal{A}_{\boldsymbol{w}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^{k+1} \\ p^{k+1} \\ \boldsymbol{w}^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -\alpha \mathcal{B}_{\boldsymbol{u}} & \frac{\alpha^2}{\lambda + \frac{2\mu}{d}} \mathcal{I}_p & 0 \\ 0 & \tau \mathcal{B}_{\boldsymbol{w}}^T & \tau^2 \frac{\lambda + \frac{2\mu}{d}}{\alpha^2} \mathcal{A}_{\boldsymbol{w}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^{k} \\ p^{k} \\ \boldsymbol{w}^{k} \end{pmatrix} + \begin{pmatrix} f_{\boldsymbol{u}} \\ f_p \\ f_{\boldsymbol{w}} \end{pmatrix}$$

# Square Domain Problem with Uniform Load

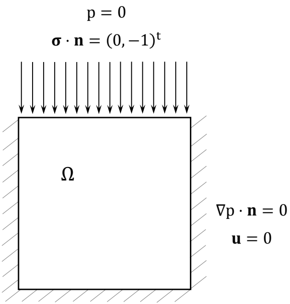


Figure: Computational domain and boundary conditions

- Uniform porous material
- Apply a uniform load on the top
- Impermeable and rigid on the other sides
- Tolerance of Krylov iterative method is $10^{-6}$

# Performance of Preconditioner: $K = 10^{-6}$ and $\nu = 0.2$

(1) Preconditioner $\mathbf{B}_D$

| $\tau$ \ $h$ | 1/8 | 1/16 | 1/32 | 1/64 |
|---|---|---|---|---|
| 0.1 | 42 | 46 | 48 | 52 |
| 0.01 | 41 | 43 | 47 | 47 |
| 0.001 | 41 | 43 | 43 | 47 |
| 0.0001 | 41 | 43 | 43 | 46 |

(2) Preconditioner $\mathbf{B}_L$

| $\tau$ \ $h$ | 1/8 | 1/16 | 1/32 | 1/64 |
|---|---|---|---|---|
| 0.1 | 22 | 23 | 24 | 25 |
| 0.01 | 22 | 22 | 23 | 25 |
| 0.001 | 22 | 22 | 23 | 24 |
| 0.0001 | 22 | 22 | 23 | 24 |

(3) Preconditioner $\mathbf{B}_U$

| $\tau$ \ $h$ | 1/8 | 1/16 | 1/32 | 1/64 |
|---|---|---|---|---|
| 0.1 | 23 | 24 | 26 | 27 |
| 0.01 | 20 | 21 | 23 | 24 |
| 0.001 | 18 | 18 | 21 | 22 |
| 0.0001 | 18 | 18 | 19 | 19 |

(4) Preconditioner $\mathbf{M}_D$

| $\tau$ \ $h$ | 1/8 | 1/16 | 1/32 | 1/64 |
|---|---|---|---|---|
| 0.1 | 42 | 48 | 54 | 57 |
| 0.01 | 42 | 44 | 49 | 50 |
| 0.001 | 42 | 44 | 49 | 49 |
| 0.0001 | 42 | 46 | 49 | 50 |

(5) Prconditioner $\mathbf{M}_L$

| $\tau$ \ $h$ | 1/8 | 1/16 | 1/32 | 1/64 |
|---|---|---|---|---|
| 0.1 | 24 | 26 | 26 | 31 |
| 0.01 | 24 | 25 | 27 | 28 |
| 0.001 | 24 | 25 | 27 | 28 |
| 0.0001 | 24 | 25 | 27 | 28 |

(6) Preconditioner $\mathbf{M}_U$

| $\tau$ \ $h$ | 1/8 | 1/16 | 1/32 | 1/64 |
|---|---|---|---|---|
| 0.1 | 26 | 30 | 29 | 32 |
| 0.01 | 24 | 26 | 26 | 28 |
| 0.001 | 25 | 24 | 24 | 26 |
| 0.0001 | 25 | 29 | 30 | 30 |

# Performance of Preconditioners: Varying $K$ and $\nu$

Table: Varying $K$: $\nu = 0.2$, $h = 1/32$ and $\tau = 0.01$

| | 1 | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ | $10^{-8}$ | $10^{-10}$ |
|---|---|---|---|---|---|---|
| $\mathbf{B}_D$ | 80 | 61 | 47 | 47 | 47 | 47 |
| $\mathbf{B}_L$ | 54 | 41 | 23 | 23 | 23 | 23 |
| $\mathbf{B}_U$ | 52 | 41 | 23 | 23 | 23 | 23 |
| $\mathbf{M}_D$ | 98 | 63 | 49 | 49 | 49 | 49 |
| $\mathbf{M}_L$ | 76 | 45 | 27 | 27 | 27 | 27 |
| $\mathbf{M}_U$ | 68 | 44 | 30 | 30 | 30 | 30 |

Table: Varying $\nu$: $K = 10^{-6}$, $h = 1/32$ and $\tau = 0.01$

| | 0.1 | 0.2 | 0.4 | 0.45 | 0.49 | 0.499 |
|---|---|---|---|---|---|---|
| $\mathbf{B}_D$ | 51 | 47 | 30 | 24 | 17 | 13 |
| $\mathbf{B}_L$ | 26 | 23 | 15 | 12 | 9 | 7 |
| $\mathbf{B}_U$ | 26 | 23 | 16 | 13 | 9 | 6 |
| $\mathbf{M}_D$ | 53 | 49 | 35 | 29 | 24 | 21 |
| $\mathbf{M}_L$ | 30 | 27 | 19 | 16 | 14 | 13 |
| $\mathbf{M}_U$ | 33 | 30 | 22 | 17 | 18 | 20 |

# Maxwell's System

Let $\mathcal{O}$ be a bounded and connected domain, we consider the Maxwell's equation in the exterior of $\bar{\mathcal{O}}$, i.e. $\Omega = \mathbb{R}^3 \backslash \bar{\mathcal{O}}$

$$
\begin{aligned}
\boldsymbol{B}_t + \operatorname{curl} \boldsymbol{E} &= 0 && \text{[Faraday's Law ]}\\
\varepsilon \boldsymbol{E}_t - \operatorname{curl} \mu^{-1} \boldsymbol{B} &= -\boldsymbol{j} && \text{[Ampere's Law]}\\
\operatorname{div} \varepsilon \boldsymbol{E} &= \boldsymbol{\rho} && \text{[Gauß's law]}\\
\operatorname{div} \boldsymbol{B} &= 0 && \text{[Solenoidal constraint]}
\end{aligned}
$$

$\varepsilon$ - permitivity of the medium; $\mu$ - magnetic permeability

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$\varepsilon$ - permitivity of the medium; $\mu$ - magnetic permeability

Remark: computationally, we use $\Omega = \mathcal{S} \backslash \bar{\mathcal{O}}$ where $\mathcal{S}$ is a ball in $\mathbb{R}^3$ with sufficiently large radius

# Boundary Conditions

# Boundary Conditions

**Perfect Conductor Boundary Conditions on the Outer Sphere $\mathcal{S}$**

$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0}$ and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, where $\boldsymbol{n}$ is the outward unit normal

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**Impedance/Dissipative Boundary Conditions on $\Gamma_i = \partial\Omega \cap \partial\mathcal{O}$**

$(1+\gamma)\boldsymbol{E}_{\tan} = -\boldsymbol{n} \times \boldsymbol{B}_{\tan}$ or equivalently $(1+\gamma)\boldsymbol{E}_{\tan} = -\boldsymbol{n} \times \boldsymbol{B}$

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Remarks:

- Many materials are not perfect conductors, but allow the fields to penetrate only a small distance
- Notation: $\boldsymbol{E}_{\tan} := \boldsymbol{E} - (\boldsymbol{n} \cdot \boldsymbol{E})\ \boldsymbol{n}$

# Asymptotically Disappearing Solutions

## Asymptotically Disappearing Solutions (ADS)

Solutions to Maxwell's equations, whose total energy decays exponentially with time.

Ref: F. Colombini, V. Petkov, and J. Rauch, *Incoming and disappearing solutions for Maxwell's equations*, Proc. Amer. Math. Soc., 139 (2011).

# Asymptotically Disappearing Solutions

## Asymptotically Disappearing Solutions (ADS)

Solutions to Maxwell's equations, whose total energy decays exponentially with time.

- Dissipative systems (boundary conditions) are of interest because not everything is perfect conductor.
- The existence of ADS shows that the leading term of the back-scattering operator is negligibly small and indicates loss of information. (Majda, 1976)
- Shown by Colombini, Petkov, and Rauch that with appropriate dissipative boundary conditions on the exterior of a sphere, ADS are obtained.

Ref: F. Colombini, V. Petkov, and J. Rauch, *Incoming and disappearing solutions for Maxwell's equations*, Proc. Amer. Math. Soc., 139 (2011).

# Asymptotically Disappearing Solutions

## Asymptotically Disappearing Solutions

The function $u = (E, B)$ is called asymptotically disapearing solution if

$$u(t, x) = e^{\lambda t} w(x),$$

with $\mathfrak{Re}(\lambda) < 0$.

- For the sphere (and other domains with smooth boundary), the spectra of the semi-group generator corresponding to the dissipative b.c. has isolated eigenvalues with $\mathfrak{Re}(\lambda) < 0$. They are all of finite multiplicity.
- The dimension of the space spanned by the generalized eigenfunctions corresponding to these eigenvalues, $\lambda$, is also finite.
- The corresponding eigenfunctions, when taken as initial conditions, lead to asymptotically disappearing solutions.

# Variational Weak Form

# Variational Weak Form

- Introduce an auxiliary variable $p$ associated with the constraint of $\boldsymbol{E}$
- $H_{\mathrm{imp}}(\mathrm{div}) = \{\boldsymbol{v} \in H(\mathrm{div}) \text{ such that } \boldsymbol{v} \cdot \boldsymbol{n}\big|_{\partial\Omega\setminus\Gamma_i} = 0\}$
- $H_{\mathrm{imp}}(\mathrm{curl}) = \{\boldsymbol{v} \in H(\mathrm{curl}) \text{ such that } \boldsymbol{v} \times \boldsymbol{n}\big|_{\partial\Omega\setminus\Gamma_i} = 0\}$
- $H_0(\mathrm{grad}) = \{v \in H^1(\Omega) \text{ such that } v\big|_{\partial\Omega} = 0\}$

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- $H_0(\mathrm{grad}) = \{v \in H^1(\Omega) \text{ such that } v\big|_{\partial\Omega} = 0\}$

Weak Form: Find $(\boldsymbol{B}, \boldsymbol{E}, p) \in H_{\mathrm{imp}}(\mathrm{div}; t) \times H_{\mathrm{imp}}(\mathrm{curl}; t) \times H_0(\mathrm{grad}; t)$, $\forall$ $(\boldsymbol{C}, \boldsymbol{F}, q) \in H_{\mathrm{imp}}(\mathrm{div}) \times H_{\mathrm{imp}}(\mathrm{curl}) \times H_0^1(\Omega)$ and for all $t > 0$,

$$\langle \mu^{-1}\boldsymbol{B}_t, \boldsymbol{C}\rangle + \langle \mu^{-1}\,\mathrm{curl}\ \boldsymbol{E}, \boldsymbol{C}\rangle = 0$$

$$\langle \varepsilon\boldsymbol{E}_t, \boldsymbol{F}\rangle + \langle \varepsilon\ \mathrm{grad}\ \ p, \boldsymbol{F}\rangle - \langle \mu^{-1}\boldsymbol{B}, \mathrm{curl}\ \boldsymbol{F}\rangle + (1+\gamma)\int_{\Gamma_i} \langle \boldsymbol{E}_{\tan}, \boldsymbol{F}_{\tan}\rangle = -(\boldsymbol{j}, \boldsymbol{F})$$

$$\langle p_t, q\rangle - \langle \varepsilon\boldsymbol{E}, \mathrm{grad}\ \ q\rangle = 0$$

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$$\langle p_t, q\rangle - \langle \varepsilon\boldsymbol{E}, \text{grad }\; q\rangle = 0$$

**Proposition** [Adler, Petkov, & Z. 2013]

- If $\text{div}\,\boldsymbol{B}_0 = 0$, then $\text{div}\,\boldsymbol{B} = 0$ for all $t > 0$
- If $\langle \boldsymbol{E}_0, \text{grad}\, q\rangle = 0$, then $\langle \boldsymbol{E}, \text{grad}\, q\rangle = 0$, $\forall q \in H_0(\text{grad})$, for all $t > 0$

# Structure-Preserving Discretization

- $\boldsymbol{B}_h^n \in H_{h,\mathrm{imp}}(\mathrm{div}) \subset H_{\mathrm{imp}}(\mathrm{div})$: Raviart-Thomas element
- $\boldsymbol{E}_h^n \in H_{h,\mathrm{imp}}(\mathrm{curl}) \subset H_{\mathrm{imp}}(\mathrm{curl})$: Nédélec element
- $p_h^n \in H_{h,0}(\mathrm{grad}) \subset H_0(\mathrm{grad})$: linear element

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Full Discretization: Crank-Nicolson scheme for temporal discretization

$$\langle \mu^{-1} \frac{\boldsymbol{B}_h^n - \boldsymbol{B}_h^{n-1}}{\tau}, \boldsymbol{C}_h \rangle + \langle \mu^{-1} \,\mathrm{curl}\, \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2}, \boldsymbol{C}_h \rangle = 0$$

$$\langle \varepsilon \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau}, \boldsymbol{F}_h \rangle + \langle \varepsilon \frac{\mathrm{grad}\, p_h^n + \mathrm{grad}\ p_h^{n-1}}{2}, \boldsymbol{F}_h \rangle - \langle \mu^{-1} \frac{\boldsymbol{B}_h^n + \boldsymbol{B}_h^{n-1}}{2}, \mathrm{curl}\, \boldsymbol{F}_h \rangle$$

$$+(1+\gamma) \int_{\Gamma_i} \langle \frac{\boldsymbol{E}_{h,\tan}^n + \boldsymbol{E}_{h,\tan}^{n-1}}{2}, \boldsymbol{F}_{h,\tan} \rangle = -( \frac{\boldsymbol{j}^n + \boldsymbol{j}^{n-1}}{2}, \boldsymbol{F}_h)$$

$$\langle \frac{p_h^n - p_h^{n-1}}{\tau}, q_h \rangle - \langle \varepsilon \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2}, \mathrm{grad}\, q_h \rangle = 0$$

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$$\langle \frac{p_h^n - p_h^{n-1}}{\tau}, q_h \rangle - \langle \varepsilon \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2}, \mathrm{grad}\, q_h \rangle = 0$$

**Proposition** [Adler, Petkov, & Z. 2013]

- If $\mathrm{div}\, \boldsymbol{B}_h^0 = 0$, then $\mathrm{div}\, \boldsymbol{B}_h^n = 0$ for all $n \geq 1$
- If $\langle \boldsymbol{E}_h^0, \mathrm{grad}\, q \rangle = 0$, then $\langle \boldsymbol{E}_h^n, \mathrm{grad}\, q \rangle = 0$, $\forall q \in H_{h,0}(\mathrm{grad})$, for all $n \geq 1$

# Poincaré-Steklov operator definition of ADS

- ADS: $\boldsymbol{j} = 0$, and $(\boldsymbol{E}, \boldsymbol{B}) \leftarrow e^{-\eta t}(\boldsymbol{E}, \boldsymbol{B})$, with $\mathfrak{Re}(\eta) > 0$.
- From the weak form we have: $\boldsymbol{B} = \eta^{-1}\,\mathrm{curl}\,\boldsymbol{E}$ and taking $F = \nabla q$ in the second equation gives $p = 0$.
- Thus, $(\boldsymbol{E}, \boldsymbol{B} = \eta^{-1}\,\mathrm{curl}\,\boldsymbol{E})$ is an ADS iff

$$\langle A(\eta)\boldsymbol{E}, \boldsymbol{F}\rangle = \eta\langle \boldsymbol{E}, \boldsymbol{F}\rangle + \frac{1}{\eta}\langle \mathrm{curl}\,\boldsymbol{E}, \mathrm{curl}\,\boldsymbol{F}\rangle = (1+\gamma)\langle E_{\tan}, F_{\tan}\rangle_{\Gamma_i}.$$

- Let $S(\eta)$ be the Poincaré-Steklov operator for $A(\eta)$, i.e. the Schur complement corresponding to "interior/boundary" splitting.
- Action of $S(\eta)$, i.e. $S(\eta)\boldsymbol{g}$ for a given $\boldsymbol{g}$ (on the boundary):
  - Solve $A(\eta)\boldsymbol{E} = \boldsymbol{0}$, in $\Omega$ and $\boldsymbol{E}_{\tan} = \boldsymbol{g}$ on $\Gamma_i$.
  - Define $\langle S(\eta)\boldsymbol{g}, \boldsymbol{w}\rangle_{\Gamma_i} := \langle A(\eta)\boldsymbol{E}, \widetilde{\boldsymbol{w}}\rangle$, for all $\boldsymbol{w}$ in some trace space.
  - Here, for $\boldsymbol{w}$ given on $\Gamma_i$, $\widetilde{\boldsymbol{w}}$ is an $\boldsymbol{H}(\mathrm{curl})$ bdd extension of $\boldsymbol{w}$ in $\Omega$.

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- Let $S(\eta)$ be the Poincaré-Steklov operator for $A(\eta)$, i.e. the Schur complement corresponding to "interior/boundary" splitting.

**ADS definition:**

Given $\gamma$, find $\eta \in \mathbb{C}$, such that $(1+\gamma)$ is an eigenvalue of $S(\eta)$. The corresponding eigenvector is $\boldsymbol{E}_{\tan}$ which determines $\boldsymbol{E}$ uniquelly.

- Non-linear (inverse) eigenvalue problem and its solution by Beyn's method based on contour integration (Beyn 2012) requires **repeated** computation of the action of $S(\eta)$.

# Difficulties in modeling ADS:

- Time dependent problem: Solving the linear system on every time step;
- Poincaré-Steklov formulation: Calculating just one the action of $S(\eta)$ requires solution of the discretized Maxwell's system.
- Efficient Solver is needed!!

## For numerical modelling of ADS:

- Design efficient solvers for solving linear systems at each time step
- Operator preconditioners based on exact block factorization
- Show robustness with respect to the physical and discretization parameters

# Linear systems modeling ADS

Linear system: Find $(\boldsymbol{B}, \boldsymbol{E}, p) \in H_{h,\mathrm{imp}}(\mathrm{div}) \times H_{h,\mathrm{imp}}(\mathrm{curl}) \times H_{h,0}(\mathrm{grad})$, such that for all $(\boldsymbol{C}, \boldsymbol{F}, q) \in H_{h,\mathrm{imp}}(\mathrm{div}) \times H_{h,\mathrm{imp}}(\mathrm{curl}) \times H_{h,0}(\mathrm{grad})$,

$$\frac{2}{\tau}\langle \mu^{-1}\boldsymbol{B}, \boldsymbol{C}\rangle + \langle \mu^{-1}\,\mathrm{curl}\,\boldsymbol{E}, \boldsymbol{C}\rangle = \langle \boldsymbol{g_B}, \boldsymbol{C}\rangle$$

$$\frac{2}{\tau}\langle \varepsilon\boldsymbol{E}, \boldsymbol{F}\rangle + \langle \varepsilon\,\mathrm{grad}\,p, \boldsymbol{F}\rangle - \langle \mu^{-1}\boldsymbol{B}, \mathrm{curl}\,\boldsymbol{F}\rangle + (1+\gamma)\int_{\Gamma_i}\langle \boldsymbol{E}_{\tan}, \boldsymbol{F}_{\tan}\rangle = (\boldsymbol{g_E}, \boldsymbol{F})$$

$$\frac{2}{\tau}\langle p, q\rangle - \langle \varepsilon\boldsymbol{E}, \mathrm{grad}\,q\rangle = \langle g_p, q\rangle$$

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$$\frac{2}{\tau}\langle \varepsilon\boldsymbol{E}, \boldsymbol{F}\rangle + \langle \varepsilon\,\mathrm{grad}\,p, \boldsymbol{F}\rangle - \langle \mu^{-1}\boldsymbol{B}, \mathrm{curl}\,\boldsymbol{F}\rangle + (1+\gamma)\int_{\Gamma_i}\langle \boldsymbol{E}_{\tan}, \boldsymbol{F}_{\tan}\rangle = (\boldsymbol{g_E}, \boldsymbol{F})$$

$$\frac{2}{\tau}\langle p, q\rangle - \langle \varepsilon\boldsymbol{E}, \mathrm{grad}\,q\rangle = \langle g_p, q\rangle$$

Matrix form:

$$\mathcal{A}\boldsymbol{x} = \boldsymbol{F} \iff \begin{pmatrix} \frac{2}{\tau}\mathcal{M}_f & \mathcal{M}_f\mathcal{K}_{fe} & \\ -\mathcal{K}_{fe}^T\mathcal{M}_f & \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z} & \mathcal{M}_e\mathcal{G}_{ev} \\ & -\mathcal{G}_{ev}^T\mathcal{M}_e & \frac{2}{\tau}\mathcal{M}_v \end{pmatrix} \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{E} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{g_B} \\ \boldsymbol{g_E} \\ g_p \end{pmatrix}.$$

- $\mathcal{M}_f$, $\mathcal{M}_e$, and $\mathcal{M}_v$ are the mass matrices of Raviart-Thomas, Nédélec, and linear elements
- $\mathcal{G}_{ev}$ and $\mathcal{K}_{fv}$ are the edge-to-vertex and face-to-edge incidence matrices
- $\mathcal{Z}$ is the matrix associated with the impedance boundary condition

# Well-posedness

Design robust preconditioners: $\mathcal{A}\boldsymbol{x} = \boldsymbol{F}$ and $\mathcal{A} : \mathcal{H} \mapsto \mathcal{H}'$, $\mathcal{H}$ is a Hilbert space

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**Theorem (Adler, Hu, & Z. 2016)**

*The linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$ and the constants involved do not depend on the physical parameters $\mu$, $\varepsilon$ and the discretization parameters $h$, $\tau$.*

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- How about computational cost? Inverting the diagonal blocks is expensive

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$$\mathcal{B}_D = \begin{pmatrix} \left(\frac{2}{\tau}\mathcal{M}_f\right)^{-1} & 0 & 0 \\ 0 & \left(\frac{\tau}{2}\mathcal{K}_{fe}^T\mathcal{M}_f\mathcal{K}_{fe} + \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z}\right)^{-1} & 0 \\ 0 & 0 & \left(\frac{\tau}{2}\mathcal{G}_{ev}^T\mathcal{M}_v\mathcal{G}_{ev} + \frac{2}{\tau}\mathcal{M}_v\right)^{-1} \end{pmatrix}$$

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Make it practical: approximate each diagonal block

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Implementation issues:

- $\mathcal{Q}_p$: algebraic multigrid (AMG); $\mathcal{Q}_{\boldsymbol{E}}$: Hiptmair-Xu (HX) preconditioner
- No need to compute $\mathcal{M}_f^{-1}$, as seen below:

$$\boldsymbol{v}_{\boldsymbol{B}}^m = \left(\frac{\tau}{2}\mathcal{M}_f\right)^{-1}\left(\frac{\tau}{2}\mathcal{M}_f\boldsymbol{v}_{\boldsymbol{B}}^{m-1} + \mathcal{M}_f\mathcal{K}_{fe}\boldsymbol{v}_{\boldsymbol{B}}^{m-1}\right) = \boldsymbol{v}_{\boldsymbol{B}}^{m-1} + \frac{2}{\tau}\mathcal{K}_{fe}\boldsymbol{v}_{\boldsymbol{B}}^{m-1}$$

# Block Triangular Preconditioners for ADS

Block triangular preconditioner: Riesz operator + lower/upper triangular

$$\mathcal{B}_L = \begin{pmatrix} \frac{2}{\tau}\mathcal{M}_f & 0 & 0 \\ -\mathcal{K}_{fe}^T\mathcal{M}_f & \frac{\tau}{2}\mathcal{K}_{fe}^T\mathcal{M}_f\mathcal{K}_{fe} + \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z} & 0 \\ 0 & -\mathcal{G}_{ev}^T\mathcal{M}_e & \frac{\tau}{2}\mathcal{G}_{ev}^T\mathcal{M}_v\mathcal{G}_{ev} + \frac{2}{\tau}\mathcal{M}_v \end{pmatrix}^{-1}$$

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- Reduce cost: approximate diagonal blocks

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**Theorem (Adler, Hu, & Z. 2016)**

*If the linear system is well-posed, we have the following Field-of-Values equivalence:*

$$\gamma \le \frac{\langle \boldsymbol{x}, \mathcal{B}_L\mathcal{A}\boldsymbol{x}\rangle_{\mathcal{B}_D}}{\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\mathcal{B}_D}}, \quad \frac{\|\mathcal{B}_L\mathcal{A}\boldsymbol{x}\|_{\mathcal{B}_D}}{\|\boldsymbol{x}\|_{\mathcal{B}_D}} \le \Gamma,$$

*which implies preconditioned GMRes converges uniformly. Moreover, PGMRes also preserves* $\operatorname{div} \boldsymbol{B}^\ell = 0$ *for* $\ell = 1, 2, \cdots$.

# Block Factroziation

Matrix in the block form:

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In general, we can factorize the block matrix: $\mathcal{A} = \mathcal{LDU}$

- Schur complements in $\mathcal{D}$ are defined recursively and usually dense
- $\mathcal{L}$ and $\mathcal{U}$ contain the inverses of Schur complements, which usually are difficult to compute
- Sparse and good approximations of the Schur complements are not easy to find

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|---|---|
| | |

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| $\operatorname{curl}\operatorname{grad} = 0$<br>$(1+\gamma)\int_{\Gamma_i}\langle \boldsymbol{n}\wedge\boldsymbol{E}, \boldsymbol{n}\wedge \operatorname{grad}\, p\rangle = 0$ | $\mathcal{K}_{fe}\mathcal{G}_{ev} = 0$ or $\mathcal{G}_{ev}^T\mathcal{K}_{fe}^T = 0$ |

# Exact Factorization of Structure-Preserving Discretization

Structure-preserving discretization:

| Continuous level | Discrete level |
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| $\operatorname{curl}\operatorname{grad} = 0$ | $\mathcal{K}_{fe}\mathcal{G}_{ev} = 0$ or $\mathcal{G}_{ev}^T\mathcal{K}_{fe}^T = 0$ |
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Exact Block Factorization: $\mathcal{A} = \mathcal{L}\mathcal{D}\mathcal{U}$

$$\mathcal{L} = \begin{pmatrix} \mathcal{I} & & \\ -\frac{\tau}{2}\mathcal{K}_{fe}^T & \mathcal{I} & \\ & -\frac{\tau}{2}\mathcal{G}_{ev}^T & \mathcal{I} \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} \frac{2}{\tau}\mathcal{M}_f & & \\ & \mathcal{S}_{\boldsymbol{E}} & \\ & & \mathcal{S}_p \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} \mathcal{I} & \frac{\tau}{2}\mathcal{K}_{fe} & \\ & \mathcal{I} & \frac{\tau}{2}\mathcal{G}_{ev} \\ & & \mathcal{I} \end{pmatrix}$$

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where

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$$\mathcal{S}_{\boldsymbol{E}} = \frac{\tau}{2}\mathcal{K}_{fe}^T\mathcal{M}_f\mathcal{K}_{fe} + \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z} \quad \Longleftrightarrow \quad \operatorname{curl}\mu^{-1}\operatorname{curl} + \boldsymbol{I}$$

$$\mathcal{S}_p = \frac{\tau}{2}\mathcal{G}_{ev}^T\mathcal{M}_v\mathcal{G}_{ev} + \frac{2}{\tau}\mathcal{M}_v \quad \Longleftrightarrow \quad \operatorname{div}\varepsilon\operatorname{grad} + \boldsymbol{I}$$

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Remark: thanks to the structure-preserving discretization

- Schur complements $\mathcal{S}_{\boldsymbol{E}}$ and $\mathcal{S}_p$ are computed exactly and they are sparse
- We can find good approximations to the Schur complements

# Block Preconditioners based on Exact Factorization

Spectral equivalent approximation of $\mathcal{D}^{-1}$: $\mathcal{Q} = \operatorname{diag}(\mathcal{Q}_{\boldsymbol{B}}, \mathcal{Q}_{\boldsymbol{E}}, \mathcal{Q}_p)$

$$c_1 \langle \mathcal{Q}\, \boldsymbol{x}, \boldsymbol{x} \rangle \leq \langle \mathcal{D}^{-1}\, \boldsymbol{x}, \boldsymbol{x} \rangle \leq c_2 \langle \mathcal{Q}\, \boldsymbol{x}, \boldsymbol{x} \rangle$$

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**Theorem** (Adler, Hu, & Z. 2016)

*If $\mathcal{Q}$ is spectrally equivalent to $\mathcal{D}$, we have*

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- $\mathcal{Q}_{\boldsymbol{B}}$: Jacobi; $\mathcal{Q}_{\boldsymbol{E}}$: HX-preconditioner; $\mathcal{Q}_p$: AMG

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- $\mathcal{Q}_{\boldsymbol{B}}$: Jacobi; $\mathcal{Q}_{\boldsymbol{E}}$: HX-preconditioner; $\mathcal{Q}_p$: AMG
- Keep ***B*** solenoidal: $\mathcal{Q}_{\boldsymbol{B}} = \left(\frac{2}{\tau}\mathcal{M}_f\right)^{-1}$

# Relation Between Two Preconditioners

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Diagonal case: two approaches are equivalent, i.e. $\mathcal{B}_D = \mathcal{M}_D$

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Triangular case: use lower triangular as an example

$$\mathcal{B}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\mathcal{K}_{fe}^T \mathcal{M}_f & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\mathcal{G}_{ev}^T \mathcal{M}_e & \mathcal{Q}_p^{-1} \end{pmatrix} \text{ and } \mathcal{M}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\frac{\tau}{2} \mathcal{K}_{fe}^T \mathcal{Q}_{\boldsymbol{B}}^{-1} & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\frac{\tau}{2} \mathcal{G}_{ev}^T \mathcal{Q}_{\boldsymbol{E}}^{-1} & \mathcal{Q}_p^{-1} \end{pmatrix}$$

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$$\mathcal{B}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\mathcal{K}_{fe}^T \mathcal{M}_f & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\mathcal{G}_{ev}^T \mathcal{M}_e & \mathcal{Q}_p^{-1} \end{pmatrix} \text{ and } \mathcal{M}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\frac{\tau}{2}\mathcal{K}_{fe}^T \mathcal{Q}_{\boldsymbol{B}}^{-1} & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\frac{\tau}{2}\mathcal{G}_{ev}^T \mathcal{Q}_{\boldsymbol{E}}^{-1} & \mathcal{Q}_p^{-1} \end{pmatrix}$$

- If $\mathcal{Q}_B^{-1} = \frac{2}{\tau}\mathcal{M}_f$ and $\mathcal{Q}_{\boldsymbol{E}}^{-1} = \mathcal{S}_{\boldsymbol{E}}$,

$$-\frac{\tau}{2}\mathcal{K}_{fe}^T \mathcal{Q}_{\boldsymbol{B}}^{-1} = -\frac{\tau}{2}\mathcal{K}_{fe}^T \left(\frac{2}{\tau}\mathcal{M}_f\right) = -\mathcal{K}_{fe}^T \mathcal{M}_f$$

$$-\frac{\tau}{2}\mathcal{G}_{ev}^T \mathcal{Q}_{\boldsymbol{E}}^{-1} = -\frac{\tau}{2}\mathcal{G}_{ev}^T \mathcal{S}_{\boldsymbol{E}} = -\frac{\tau}{2}\mathcal{G}_{ev}^T \left(\frac{\tau}{2}\mathcal{K}_{fe}^T \mathcal{M}_f \mathcal{K}_{fe} + \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z}\right) = -\mathcal{G}_{ev}^T \mathcal{M}_e$$

# Relation Between Two Preconditioners

Diagonal case: two approaches are equivalent, i.e. $\mathcal{B}_D = \mathcal{M}_D$

Triangular case: use lower triangular as an example

$$\mathcal{B}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\mathcal{K}_{fe}^T \mathcal{M}_f & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\mathcal{G}_{ev}^T \mathcal{M}_e & \mathcal{Q}_p^{-1} \end{pmatrix} \text{ and } \mathcal{M}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\frac{\tau}{2}\mathcal{K}_{fe}^T \mathcal{Q}_{\boldsymbol{B}}^{-1} & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\frac{\tau}{2}\mathcal{G}_{ev}^T \mathcal{Q}_{\boldsymbol{E}}^{-1} & \mathcal{Q}_p^{-1} \end{pmatrix}$$

- If $\mathcal{Q}_B^{-1} = \frac{2}{\tau}\mathcal{M}_f$ and $\mathcal{Q}_{\boldsymbol{E}}^{-1} = \mathcal{S}_{\boldsymbol{E}}$,

$$-\frac{\tau}{2}\mathcal{K}_{fe}^T \mathcal{Q}_{\boldsymbol{B}}^{-1} = -\frac{\tau}{2}\mathcal{K}_{fe}^T \left(\frac{2}{\tau}\mathcal{M}_f\right) = -\mathcal{K}_{fe}^T \mathcal{M}_f$$

$$-\frac{\tau}{2}\mathcal{G}_{ev}^T \mathcal{Q}_{\boldsymbol{E}}^{-1} = -\frac{\tau}{2}\mathcal{G}_{ev}^T \mathcal{S}_{\boldsymbol{E}} = -\frac{\tau}{2}\mathcal{G}_{ev}^T \left(\frac{\tau}{2}\mathcal{K}_{fe}^T \mathcal{M}_f \mathcal{K}_{fe} + \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z}\right) = -\mathcal{G}_{ev}^T \mathcal{M}_e$$

then $\mathcal{B}_L = \mathcal{M}_L$

# Relation Between Two Preconditioners

Diagonal case: two approaches are equivalent, i.e. $\mathcal{B}_D = \mathcal{M}_D$

Triangular case: use lower triangular as an example

$$\mathcal{B}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\mathcal{K}_{fe}^T\mathcal{M}_f & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\mathcal{G}_{ev}^T\mathcal{M}_e & \mathcal{Q}_p^{-1} \end{pmatrix} \text{ and } \mathcal{M}_L^{-1} = \begin{pmatrix} \mathcal{Q}_{\boldsymbol{B}}^{-1} & 0 & 0 \\ -\frac{\tau}{2}\mathcal{K}_{fe}^T\mathcal{Q}_{\boldsymbol{B}}^{-1} & \mathcal{Q}_{\boldsymbol{E}}^{-1} & 0 \\ 0 & -\frac{\tau}{2}\mathcal{G}_{ev}^T\mathcal{Q}_{\boldsymbol{E}}^{-1} & \mathcal{Q}_p^{-1} \end{pmatrix}$$

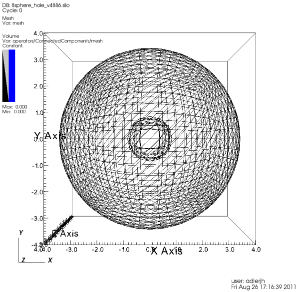
- If $\mathcal{Q}_B^{-1} = \frac{2}{\tau}\mathcal{M}_f$ and $\mathcal{Q}_{\boldsymbol{E}}^{-1} = \mathcal{S}_{\boldsymbol{E}}$,

  $$-\frac{\tau}{2}\mathcal{K}_{fe}^T\mathcal{Q}_{\boldsymbol{B}}^{-1} = -\frac{\tau}{2}\mathcal{K}_{fe}^T\left(\frac{2}{\tau}\mathcal{M}_f\right) = -\mathcal{K}_{fe}^T\mathcal{M}_f$$

  $$-\frac{\tau}{2}\mathcal{G}_{ev}^T\mathcal{Q}_{\boldsymbol{E}}^{-1} = -\frac{\tau}{2}\mathcal{G}_{ev}^T\mathcal{S}_{\boldsymbol{E}} = -\frac{\tau}{2}\mathcal{G}_{ev}^T\left(\frac{\tau}{2}\mathcal{K}_{fe}^T\mathcal{M}_f\mathcal{K}_{fe} + \frac{2}{\tau}\mathcal{M}_e + \mathcal{Z}\right) = -\mathcal{G}_{ev}^T\mathcal{M}_e$$

  then $\mathcal{B}_L = \mathcal{M}_L$
- In general, $\mathcal{B}_L$ and $\mathcal{M}_L$ are different

# Example




Parameters for the Maxwell's system: $\varepsilon = \mu^{-1} = 1$ and $\tau = 0.1$

# Performance of Preconditioners

(1) Preconditioner $\mathcal{B}_D$

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| 0.2 | 21 | 26 | 28 | 28 |
| 0.1 | 14 | 22 | 26 | 27 |
| 0.05 | 10 | 16 | 23 | 25 |
| 0.025 | 7 | 10 | 17 | 23 |

(2) Preconditioner $\mathcal{B}_L$

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| 0.2 | 7 | 8 | 10 | 10 |
| 0.1 | 6 | 7 | 8 | 9 |
| 0.05 | 5 | 6 | 7 | 8 |
| 0.025 | 5 | 5 | 6 | 7 |

(3) Preconditioner $\mathcal{B}_U$

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| 0.2 | 7 | 8 | 9 | 10 |
| 0.1 | 6 | 7 | 8 | 9 |
| 0.05 | 5 | 6 | 8 | 8 |
| 0.025 | 4 | 5 | 6 | 8 |

(4) Preconditioner $\mathcal{M}_L$

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| 0.2 | 5 | 6 | 6 | 8 |
| 0.1 | 5 | 7 | 7 | 7 |
| 0.05 | 5 | 5 | 6 | 7 |
| 0.025 | 5 | 5 | 5 | 6 |

(5) Prconditioner $\mathcal{M}_U$

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| 0.2 | 5 | 7 | 8 | 8 |
| 0.1 | 5 | 6 | 7 | 8 |
| 0.05 | 5 | 6 | 6 | 7 |
| 0.025 | 4 | 5 | 6 | 6 |

(6) Preconditioner $\mathcal{M}_{LU}$

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| 0.2 | 4 | 4 | 5 | 5 |
| 0.1 | 4 | 4 | 4 | 5 |
| 0.05 | 4 | 4 | 4 | 4 |
| 0.025 | 3 | 4 | 4 | 4 |

# Performance of Preconditioners: jumps in $\varepsilon$ and $\mu$

Table: Jumps in $\varepsilon$: $h = 1/32$ and $\tau = 0.1$

| | $10^{-6}$ | $10^{-4}$ | $10^{-2}$ | $1$ | $10^{2}$ | $10^{4}$ | $10^{6}$ |
|---|---|---|---|---|---|---|---|
| $\mathcal{B}_D$ | 28 | 28 | 27 | 26 | 27 | 22 | 16 |
| $\mathcal{B}_L$ | 9 | 9 | 9 | 8 | 8 | 8 | 8 |
| $\mathcal{B}_U$ | 9 | 9 | 8 | 8 | 8 | 8 | 8 |
| $\mathcal{M}_L$ | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| $\mathcal{M}_U$ | 7 | 7 | 7 | 7 | 6 | 6 | 6 |
| $\mathcal{M}_{LU}$ | 5 | 4 | 4 | 4 | 4 | 4 | 4 |

Table: Jumps in $\mu$: $h = 1/32$ and $\tau = 0.1$

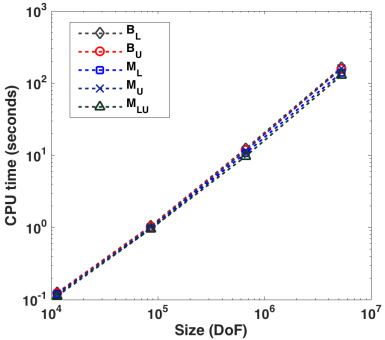
| | $10^{-6}$ | $10^{-4}$ | $10^{-2}$ | $1$ | $10^{2}$ | $10^{4}$ | $10^{6}$ |
|---|---|---|---|---|---|---|---|
| $\mathcal{B}_D$ | 17 | 22 | 27 | 26 | 26 | 26 | 26 |
| $\mathcal{B}_L$ | 11 | 11 | 9 | 8 | 8 | 8 | 8 |
| $\mathcal{B}_U$ | 10 | 10 | 9 | 8 | 8 | 8 | 8 |
| $\mathcal{M}_L$ | 9 | 8 | 7 | 7 | 7 | 7 | 7 |
| $\mathcal{M}_U$ | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| $\mathcal{M}_{LU}$ | 5 | 5 | 5 | 4 | 4 | 4 | 5 |

# Computational Complexity

Parameters: $\varepsilon = \mu^{-1} = 1$ and $\tau = 0.1$

Average CPU time (seconds) over 20 time steps

| | $1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---|---|---|---|---|
| $\mathcal{B}_L$ | 0.13 | 1.05 | 12.38 | 165.23 |
| $\mathcal{B}_U$ | 0.12 | 1.03 | 11.79 | 158.55 |
| $\mathcal{M}_L$ | 0.12 | 0.96 | 10.72 | 138.83 |
| $\mathcal{M}_U$ | 0.12 | 1.00 | 11.65 | 159.71 |
| $\mathcal{M}_{LU}$ | 0.11 | 0.95 | 9.70 | 127.80 |



# Conclusions

- We presented structure-preserving and stable finite element methods for approximate solutions to the systems of coupled PDEs (poroelasticity and Maxwell's system)
- We designed family of operator preconditioners based on the well-posedness of the discrete systems (discretized Biot's model and Maxwell's system)
- We have shown robustness of the preconditioners with respect to the physical and discretization parameters
- We also presented a novel approach for the analysis for the sequential-implicit (splitting) schemes for poroelasticity.

**Software**

All numerical experiments in this talk are done using the HAZMATH FE and graph library: `https://bitbucket.org/XiaozheHu/hazmath/wiki/Home`

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Thank You!