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Weak turbulence for the nonlinear Schrödinger equation: a connection between the scattering theory and the Arnold diffusion

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The linear periodic Shrödinger equation

• In dimension 1, the linear periodic Schrödinger equation reads

$$\partial_t u = i\,\partial_x^2 u,\ \ u(0,x) = u_0(x), \tag{1}$$

where $i = \sqrt{-1}$, $t \in \mathbb{R}$, $x \in \mathbb{T} = \mathbb{R}|(2\pi\mathbb{Z})$ and $u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$.

• For $u_0 \in C^\infty(\mathbb{T})$ the solution of (1) is given by the exponential sum

$$u(t,x) = \sum_{n\in\mathbb{Z}} e^{-itn^2}\, e^{inx}\, \widehat{u_0}(n),$$

where $\widehat{u_0}(n)$ is the n'th Fourier coefficient of $u_0(x)$, i.e.

$$\widehat{u_0}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} u_0(x)\, dx.$$

• Observe that $u(t,x)$ is 2π-periodic in time : $u(t+2\pi,x) = u(t,x)$.

Conservation of the Sobolev norms

• If a function $u : \mathbb{T} \to \mathbb{C}$ has a Fourier expansion

$$u(x) = \sum_{n \in \mathbb{Z}} e^{inx}\, \widehat{u}(n)$$

then for $s \in \mathbb{R}$, the Sobolev norm H^s of u is defined by

$$\|u\|_{H^s}^2 = \sum_{n \in \mathbb{Z}} (1+n^2)^s |\widehat{u}(n)|^2 \,.$$

For $s = 0$, we recover an equivalent to the L^2 norm. We also have

$$\|u\|_{H^1} \approx \|u\|_{L^2} + \|u'\|_{L^2}, \quad \|u\|_{H^2} \approx \|u\|_{L^2} + \|u'\|_{L^2} + \|u''\|_{L^2}, \quad \text{etc.}$$

• It is now clear that the above solution of the linear periodic Schrödinger equation satisfies

$$\|u(t,\cdot)\|_{H^s} = \|u_0\|_{H^s}, \quad \forall\, t \in \mathbb{R}. \tag{2}$$

• We can therefore uniquely extend the solution map $u_0(x) \mapsto u(t,x)$ to a continuous map from $H^s(\mathbb{T})$ to $C(\mathbb{R}; H^s(\mathbb{T}))$, $s \in \mathbb{R}$. Moreover the $H^s(\mathbb{T})$ norm is preserved, i.e. we have (2).

The linear Shrödinger equation on the line

• Consider now the $1d$ linear Schrödinger equation on the real line

$$\partial_t u = i\,\partial_x^2 u,\; u(0,x) = u_0(x) \quad t \in \mathbb{R},\; x \in \mathbb{R}. \tag{3}$$

If u_0 is in the Schwartz class then the solution is given by

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi^2}\, e^{ix\xi}\,\widehat{u_0}(\xi)d\xi,$$

where $\widehat{u_0}(\xi)$, $\xi \in \mathbb{R}$ is the Fourier transform of u_0, defined by

$$\widehat{u_0}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} u_0(x)dx\,.$$

• The Sobolev norm H^s of functions on $\mathbb{R}$ is now defined by

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{f}(\xi)|^2 d\xi\,.$$

Since

$$\widehat{u(t,\cdot)}(\xi) = e^{-it\xi^2}\,\widehat{u_0}(\xi) \Longrightarrow |\widehat{u(t,\cdot)}(\xi)| = |\widehat{u_0}(\xi)|$$

the solution of (3) satisfies

$$\|u(t,\cdot)\|_{H^s} = \|u_0\|_{H^s}\,.$$

The dispersion

- By applying a stationary phase estimate to

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi^2} e^{ix\xi} \widehat{u_0}(\xi) d\xi,$$

we obtain that there is $c \in \mathbb{C}$ and $C > 0$ such that for every $t \geq 1$, every $x \in \mathbb{R}$,

$$\left| u(t,x) - c \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \widehat{u_0}(x/2t) \right| \leq C t^{-\frac{3}{4}} \|x u_0\|_{L^2} .$$

In particular, for $t \geq 1$,

$$|u(t,x)| \leq C(u_0)\, t^{-\frac{1}{2}}, \quad \forall\, x \in \mathbb{R}.$$

- Therefore the solution disperses keeping the H^s norms conserved.
- Another manifestation of the dispersion is the Strichartz estimate

$$\|u(t,x)\|_{L^6(\mathbb{R}\times\mathbb{R})} \leq C \|u_0(x)\|_{L^2(\mathbb{R})} .$$

A fully non linear model

•Consider the equation

$$\partial_t u = -i|u|^2 u, \quad u(0,x) = u_0(x).$$

• For $u_0 \in L^2$, the solution is given by :

$$u(t,x) = e^{-it|u_0(x)|^2} u_0(x)\,.$$

• Then

$$\partial_x u(t,x) = e^{-it|u_0(x)|^2} \Big(\partial_x u_0(x) - itu_0(x)\,\partial_x(|u_0(x)|^2)\Big).$$

• Therefore for $u_0(x)$ such that $|u_0(x)|$ is not a constant, there exists $C > 0$ and $A \geq 1$ such that for $t \geq A$,

$$\|u(t,\cdot)\|_{H^1} \geq Ct,$$

i.e. the H^1 norm grows in time ! Similarly for H^s, $s \geq 0$ initial data

$$\|u(t,\cdot)\|_{H^s} \geq Ct^s.$$

The 1d Nonlinear Schrödinger equation (NLS)

• We considered so far the linear model

$$\partial_t u = i\,\partial_x^2 u$$

and the fully nonlinear model

$$\partial_t u = -i|u|^2 u$$

• The $1d$ NLS is obtained when one takes into account both effects :

$$\partial_t u = i\,\partial_x^2 u - i|u|^2 u$$

or equvalently

$$i\partial_t u + \partial_x^2 u = |u|^2 u\,.$$

• For the linear model the Sobolev noms H^s of the solutions remain bounded while for the fully nonlinear model they grow as far as $s > 0$.

• **The question we discuss today is which effect dominates in the context of NLS.**

Global well-posedness and basic conservation laws for NLS

• Thanks to the 1d Sobolev emebedding $H^1 \subset L^\infty$ which makes that in 1d the Sobolev space H^1 is an algebra, we can easily solve locally in H^1 the initial value problem for

$$i\partial_t u + \partial_x^2 u = |u|^2 u\,. \tag{4}$$

• Multiply (4) with $\bar{u}$, $i\partial_t \bar{u}$, integrate over x and take the imaginary part. It comes :

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = 0, \quad \frac{d}{dt}\Big(\|\partial_x u(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|u(t,\cdot)\|_{L^4}^4\Big) = 0\,.$$

• One can deduce the second conservation law as the Hamiltonian conservation resulting from the Hamiltonian formulation of NLS.

• Therefore, for $s \geq 1$ we can extend globally in time the local solutions. Moreover, the L^2 and the H^1 norms of the solutions remain bounded in time. Therefore, concerning the H^1 norm, the linear effect dominates.

Question : What about the H^s, $s > 1$ norms ?

Remark : The question of the growth of the Sobolev norms may be seen as a competition between the kinetic and the potential energies.

Higher order conservation laws for 1d NLS

• Using the Lax representation of the 1d NLS, Zakharov-Shabat (1972) obtained that if u is an H^s, $s \geq 2$ solution of

$$i\partial_t u + \partial_x^2 u = |u|^2 u$$

then

$$\frac{d}{dt}\Big(\|\partial_x^2 u\|_{L^2}^2 + 2\|\mathrm{Re}(\partial_x u\,\bar{u})\|_{L^2}^2 + 3\|u\partial_x u\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^6}^6\Big) = 0\,.$$

Here x can be both in $\mathbb{T}$ or $\mathbb{R}$.

• Therefore the H^2 norms of the solutions remain bounded in time.

• Similarly one gets uniform in time bounds for the H^s norms, $s = 3, 4, 5, \ldots$.

• Recent work (2016) by Koch-Tataru extends these bounds for all $s \geq 0$ in the case $x \in \mathbb{R}$ (for $x \in \mathbb{T}$, there is an earlier work by Grebert-Kappeler).

Conclusion of the 1d analysis

• In summary, for the $1d$ NLS both on $\mathbb{R}$ and $\mathbb{T}$, the linear effect dominates concerning the bounds on the Sobolev norms of the solutions. This is a consequence of the complete integrability.

• What happens in higher dimensions, i.e. for the equation

$$i\partial_t u + \Delta u = |u|^2 u,$$

where Δ is the Laplace operator ?

• **Remark.** In higher dimensions, the global well-posedness is already a quite nontrivial problem.

The $3d$ NLS

• Let (M, g) be a smooth $3d$ riemannian manifold with a Laplace-Beltrami operator Δ . Consider the Cauchy problem

$$i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times M \to \mathbb{C}. \qquad (5)$$

• As in $1d$, in the context of (5), we again have the conserved quantities

$$\|U\|_{L^2(M)}, \quad \|U\|^2_{H^1(M)} + \frac{1}{2}\|U\|^4_{L^4(M)}.$$

Theorem 1 (Burq-Gérard-Tz. 2001)

Suppose that M is compact without boundary. For $s \geq 1$ and $U_0 \in H^s(M)$ there is a unique global solution of (5) in $C(\mathbb{R}; H^s(M))$. The dependence with respect to the initial data is continuous. The L^2 and the H^1 norms of the solutions are uniformly bounded in time.

• The result remains true for non compact manifolds with a controlled behaviour at infinity such as $\mathbb{R}^3$, $\mathbb{R} \times \mathbb{T}^2$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times S^2$ or a long range perturbation of $\mathbb{R}^3$ outside a compact set.

Question : Do the H^s norms, $s \neq 0, 1$ remain bounded ?

The $3d$ NLS on $\mathbb{R}^3$

Consider the Cauchy problem

$$i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}. \tag{6}$$

Theorem 2 (Ginibre-Velo, Bourgain, Dodson)
For $s > 5/7$ the problem (6) is globally well-posed in $H^s(\mathbb{R}^3)$. Moreover, for every $U_0 \in H^s$ there is $C > 0$ such that for every $t \in \mathbb{R}$ the solution of (6) satisfies

$$\|U(t,\cdot)\|_{H^s(\mathbb{R}^3)} \leq C. \tag{7}$$

For $s \geq 1$, one may proceed in two steps :
1) Using Morawetz identities (a way of exploiting the good sign of the nonlinearity in a dispersive estimate) one first shows that the $L^p(\mathbb{R}^3)$, $p \in (2,6)$ norms of the solution go to zero as t tends to infinity.
2) Then by a perturbative analysis one reinforces this information to a control on space-time norms like $L^{10}(\mathbb{R} \times \mathbb{R}^3)$ of the solutions which in turn implies (7).

The $3d$ NLS on $\mathbb{R} \times \mathbb{T}^2$

• Consider the Cauchy problem

$$i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \to \mathbb{C}. \qquad (8)$$

1) The problem (8) is locally well-posed in $H^s(\mathbb{R} \times \mathbb{T}^2)$, $s > 1/2$ (ideas by Bourgain)
2) It is ill-posed for $s \in (0, 1/2)$ (ideas by Lebeau).
3) It is globally-well-posed for $s > 5/6$ (ideas by Tao et al.).

Theorem 3 (Pausader-Tz. 2017)
For every $s \in (1/2, \infty)$, $s \neq 1$ *there exists* $U_0 \in H^s(\mathbb{R} \times \mathbb{T}^2)$ *such that the corresponding solution of* (8) *is globally defined and*

$$\limsup_{t \to \infty} \|U(t)\|_{H^s(\mathbb{R} \times \mathbb{T}^2)} = +\infty\,.$$

• Recall that for $s \geq 1$, the conservation laws provide an a priori bound on the $H^1(\mathbb{R} \times \mathbb{T}^2)$ norm. We also always have an a priori bound on the $L^2(\mathbb{R} \times \mathbb{T}^2)$ norm. A nonlinear interpolation is therefore impossible.
• Previous work by Hani-Pausader-Tz.-Visciglia 2013, obtained this result for $s \geq 30$.

Comments

• The above result is the first rigorous result of weak wave turbulence for the Nonlinear Schrödinger equation. For $s > 1$ it represents a transfer from low to high Fourier modes. For $s \in (1/2, 1)$ it represents a transfer from high to low Fourier modes.

• This result gives a partial answer to a question posed by Bourgain in a 2000 special issue of GAFA.

• The above results may be informally formulated as :

The Nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$ is not integrable while it is on $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{R}^3$.

Reduction of the problem

• Let $U(t)$ be a solution of the cubic defocusing NLS, posed on $\mathbb{R}\times\mathbb{T}^2$. Then $F(t) = e^{-it\Delta}U(t)$ solves

$$i\partial_t F(t) = \mathcal{N}^t[F(t), F(t), F(t)],$$

where the trilinear form $\mathcal{N}^t$ is defined by

$$\mathcal{N}^t[F,G,H] := e^{-it\Delta}\Big(e^{it\Delta}F \cdot e^{-it\Delta}\overline{G} \cdot e^{it\Delta}H\Big).$$

• Denote by $\widehat{F}_p(\xi)$ or $\mathcal{F}(F)(\xi,p)$ the Fourier transform on $\mathbb{R}\times\mathbb{T}^2$ of F. Then the Fourier transform of the nonlinearity can be written as :

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,p) = \sum_{p-p_1+p_2-p_3=0} e^{it\left[|p|^2-|p_1|^2+|p_2|^2-|p_3|^2\right]} \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{F}_{p_1}(\xi-\eta)\overline{\widehat{G}_{p_2}}(\xi-\eta-\kappa)\widehat{H}_{p_3}(\xi-\kappa)\,d\kappa d\eta\,.$$

Reduction of the problem (sequel)

• Ignoring the time oscillations (normal form reduction) and a stationary phase argument ($t \gg 1$) suggests to define $\mathcal{R}$ as

$$\mathcal{F}\mathcal{R}[F,G,H](\xi,p) := \sum_{\substack{p+p_2=p_1+p_3 \\ |p|^2+|p_2|^2=|p_1|^2+|p_3|^2}} \widehat{F}_{p_1}(\xi)\overline{\widehat{G}_{p_2}}(\xi)\widehat{H}_{p_3}(\xi)$$

and one expects that the nonlinearity can be decomposed as follows

$$\mathcal{N}^t[F,G,H] = \frac{\pi}{t}\mathcal{R}[F,G,H] + \text{ better terms}$$

• We therefore define the resonant system as

$$i\partial_t G(t) = \mathcal{R}[G(t), G(t), G(t)]\,.$$

• The dependence on ξ is merely parametric.

• We prove that given a solution G of the resonant system, bounded in "some norm", there is a solution of the true problem "close" to $G(\pi \ln(t))$ for $t \gg 1$.

Reduction of the problem (sequel)

• The justification of the normal form reduction and the stationary phase is a part of the analysis typical for **the long range scattering for nonlinear PDE**.

• It is a long and involved argument, using a number of tools developed in the theory of nonlinear dispersive PDE in the last 30 years such as Bourgain/Tataru spaces and almost orthogonality arguments.

• Once this analysis is done, we are reduced to the study of the (much simpler but having a deep structure) resonant system

$$i\partial_t G(t) = \mathcal{R}[G(t), G(t), G(t)]\,.$$

Reduction to the resonant equation on $\mathbb{T}^2$

• We take initial data of the resonance system of the form

$$G_0(x,y) = \mathcal{F}_{\mathbb{R}}^{-1}(\varphi)(x)g(y), \quad x \in \mathbb{R}, y \in \mathbb{T}^2,$$

with φ real valued. The solution $G(t)$ to the resonance system with initial data $G_0(x,y)$ as above is given in Fourier space by

$$\widehat{G}_p(t,\xi) = \varphi(\xi)a_p(\varphi(\xi)^2 t), \quad a_p(0) = \mathcal{F}_{\mathbb{T}^2}(g)(p),$$

where the vector $(a_p)_{p\in\mathbb{Z}^2}$ solves the *resonant equation*

$$i\partial_t a_p(t) = \sum_{\substack{p+p_2=p_1+p_3 \\ |p|^2+|p_2|^2=|p_1|^2+|p_3|^2}} a_{p_1}(t)\overline{a_{p_2}(t)}a_{p_3}(t)\,.$$

• In particular, if $\varphi = 1$ on an open interval I, then $\widehat{G}_p(t,\xi) = a_p(t)$ for all $t \in \mathbb{R}$ and $\xi \in I$. We can therefore apply the following result.

Theorem 4 (growth for the resonant equation)
Let $s > 0$, $s \neq 1$. There exist global solutions to the resonant equation in $C(\mathbb{R}; h^s)$ such that

$$\sup_{t \geq 0} \|a_p(t)\|_{h^s} = \infty$$

but for every $\varepsilon > 0$

$$\sup_{t \geq 0} \|a_p(t)\|_{h^{s-\varepsilon}} < \infty.$$

- **Notation** :

$$\|a_p\|_{h^s}^2 := \sum_{p \in \mathbb{Z}^2} (1 + |p|^2)^s |a_p|^2.$$

- **Remark.** Unfortunately, we have that, $a_p(t) \notin h^\sigma$ for $\sigma > s$.

On the analysis of the resonant equation

• The analysis of the resonant equation is inspired by a work of Colliander-Keel-Staffilani-Takaoka-Tao.

• **It reminds the Arnold diffusion instability phenomena in Hamiltonian systems.**

• Two important aspects of the analysis are:

1) There are many invariant subspaces for the resonant equation.

2) There is a superposition principle : for some initial data it "behaves as a linear equation".

On the analysis of the resonant equation (sequel)

• It turns out that for every $N \geq 1$ there is $S_N \subset \mathbb{Z}^2$ written as a disjoint union

$$S_N = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_N,$$

with $|\Lambda_j| = 2^{N-1}$ such that S_N is invariant under the resonant equation. When j increases, the sets Λ_j have points (but not all) at larger and larger frequencies of $\mathbb{Z}^2$.

• Moreover, if we look for a solution which is constant on each Λ_j, we are reduced to study the ODE on $\mathbb{C}^N$ which writes

$$-i\dot{b}_j(t) = -|b_j(t)|^2 b_j(t) + 2b_{j-1}(t)^2\overline{b_j(t)} + 2b_{j+1}(t)^2\overline{b_j(t)},$$

$j = 1, 2, ..., N$, with the convention $b_0(t) = b_{N+1}(t) = 0$.

On the analysis of the resonant equation (sequel)

• It turns out that for every $N \geq 1$ and every $\varepsilon < 10^{-2}$ there is a solution such that

$$|b_3(0)| > 1 - \varepsilon, \quad |b_j(0)| < \varepsilon, \ j \neq 3$$

and

$$|b_{N-2}(T)| > 1 - \varepsilon, \quad |b_j(T)| < \varepsilon, \ j \neq N-2,$$

for some time $T = T(N, \varepsilon)$.

• The last result implies (only) a finite time amplification of the solutions of the NLS.

• In order to get an infinite time result one performs a suitable superposition of the above construction.