Rationality problems

Rationality

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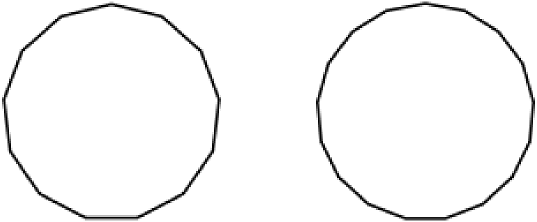
- $\sqrt{3}, \ldots, \sqrt{17}$ – Theodorus of Cyrene (400 BC?, teacher of Plato): stopped at 17, because his algebra was weak

Rationality: number theory

Classical problems:

- Doubling the cube, i.e., constructing $\sqrt[3]{2}$
- Trisecting the angle
- Constructing regular polygons

Is there a difference?



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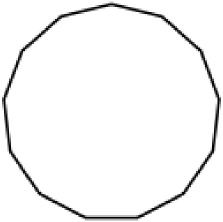

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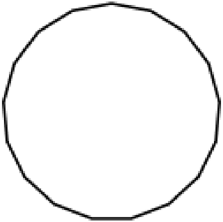

17

Gauss (1796): A regular p-gon is constructible with compass and straightedge if p is a Fermat prime, i.e., $p = 2^{2^n} + 1$; first such primes are $3, 5, 17, 257, 65537, ...$

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J. Hermes (1894): Explicit construction of the 65537-gon, 10 years of work...

Construction of regular polygons

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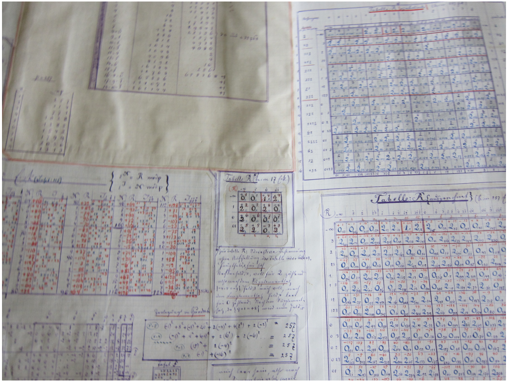

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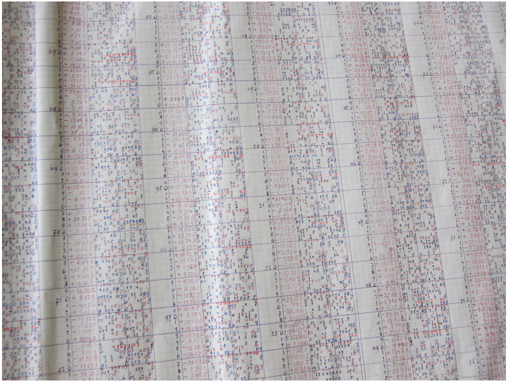

Hidden symmetries

Solutions of many classical problems, and properties of numbers, are governed by Galois groups, symmetry groups of fields.

This idea had an enormous impact on mathematics and theoretical physics.

Let K be a field, e.g., $K = \mathbb{Q}$, and

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in K[x]$$

a polynomial with coefficients in K. Let $L \subset \bar{K}$ be the smallest subfield of an algebraic closure of K containing all roots of f. The Galois group

$$\mathrm{Gal}(f) = \mathrm{Gal}(L/K) \subseteq \mathfrak{S}_n$$

is the group of automorphisms of L fixing K. It is a subgroup of the symmetric group exchanging the roots of f.

Galois groups

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is the group of automorphisms of L fixing K. It is a subgroup of the symmetric group exchanging the roots of f. It can be effectively computed, e.g.,

- $\mathrm{Gal}(x^{16} + x^{15} + \cdots + x + 1) = (\mathbb{Z}/17\mathbb{Z})^\times \simeq \mathbb{Z}/16\mathbb{Z}$, this is why the 17-gon is constructible with compass and straightedge

INVERSE GALOIS PROBLEM

Can every finite group G be realized as the Galois group of some extension of $\mathbb{Q}$?

Hilbert's irreducibility

Inverse Galois Problem

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Hilbert

Let G be a finite group. Assume that G can be realized as a Galois group of

$$K = \mathbb{Q}(x_1, \ldots, x_r).$$

Then it can be realized as the Galois group of an extension over $\mathbb{Q}$.

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Noether (1916)

Let G be a finite group, V a representation of G over a field k. Assume that $\mathbb{Q}(V)^G$ is rational. Then G can be realized as the Galois group of some extension over $\mathbb{Q}$.

Gleichungen mit vorgeschriebener Gruppe.

Von

EMMY NOETHER in Göttingen.

Das Problem der Konstruktion von Gleichungen mit vorgeschriebener Gruppe läßt sich in zwei Richtungen angreifen, die man kurz als die „irrationale" oder die Wurzeln charakterisierende, und die „rationale" oder die Koeffizienten charakterisierende, bezeichnen kann.

In der „irrationalen" Richtung, die funktionentheoretisch-arithmetisch arbeitet, liegt der Kroneckersche Satz, daß alle Abelschen Körper im Gebiet der rationalen Zahlen Kreiskörper sind, und die entsprechenden Sätze

Noether's problem

Let G be a finite group and V a representation of G over a field k. Is V/G rational?

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Let G be a finite group and V a representation of G over a field k. Is V/G rational? E.g., when k is algebraically closed?

Yes for:

- $\mathfrak{S}_n$
- abelian groups
- $\dim(V) \leq 3$

Unknown for:

- projective representations of $\mathfrak{S}_5, \mathfrak{S}_6, \mathfrak{A}_6, \mathfrak{A}_7 \hookrightarrow \mathrm{PGL}_4(\mathbb{C})$
- $\mathrm{SL}_2(\mathbb{F}_7) \subset \mathrm{SL}_4(\mathbb{C})$

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There exist groups G of order ℓ^9 and representations V such that V/G is not rational.

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There exist groups G of order ℓ^6 with V/G not rational.

- Obstruction lies in Galois cohomology $H^2_{nr}(k(V/G), \mathbb{Z}/\ell)$,
- Starting point of birational Almost abelian anabelian geometry program of Bogomolov

Rationality: algebra

A field K/k is

(R) rational: if $K \simeq k(x_0, \ldots, x_n)$ for some n, i.e., if K is a purely-transcendental extension of k

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(U) unirational: if $K \subset k(x_0, \ldots, x_n)$, for some n

A projective algebraic variety X/k "is" the set of solutions of a system of homogeneous polynomial equations with coefficients in k, e.g.,

$$x^2 + y^2 = z^2, \quad t(x^2 + y^2 + z^2) = xyz, ...$$

The standard example is projective space $\mathbb{P}^n$, when there are no equations.

Rationality: geometry

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The standard example is projective space $\mathbb{P}^n$, when there are no equations.

A fundamental problem is to determine how close an algebraic variety X is to $\mathbb{P}^n$.

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$$(R) \Rightarrow (S) \Rightarrow (U)$$

Conics

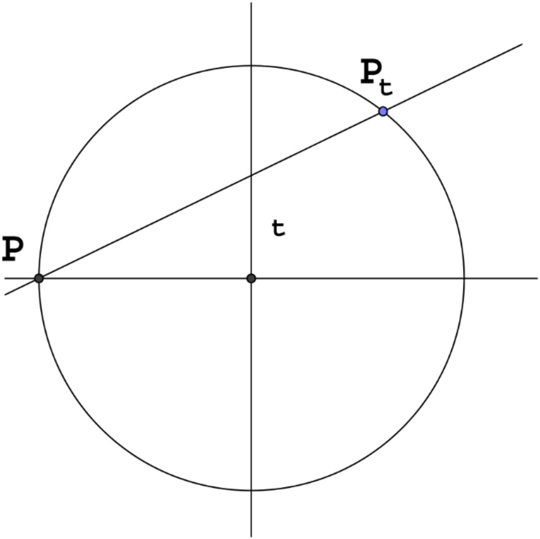
All rational solutions (rational points) of

$$x^2 + y^2 = 1$$

are given by

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}.$$

Rational curve: parametrized by rational functions in one variable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Curves: Lüroth 1876

Beweis eines Satzes über rationale Curven.

Von J. Lüroth in Karlsruhe.

Wenn die Coordinaten eines Punktes einer Curve sich darstellen lassen als rationale Functionen eines Parameters λ, so entspricht stets jedem Werth von λ nur ein Punkt der Curve, dagegen braucht nicht immer jedem Punkt der Curve nur ein Werth von λ zu entsprechen, wie das Beispiel der Gleichungen $x = \lambda^2$, $y = \frac{1}{\lambda^2}$ zeigt.

Theorem

In dimension 1, rationality = stable rationality=unirationality.

Surfaces: Castelnuovo, Enriques

Theorem

In dimension 2, over $\mathbb{C}$,

$$\textit{rationality} = \textit{stable rationality} = \textit{unirationality}$$

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Theorem

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This can fail over nonclosed ground-fields k.

Approach: via classification.

Classification schemes

- via dimension,

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- via degree (in some embedding into projective space); e.g.,

Fano, general type, intermediate type,

depending on ampleness of the (anti)canonical class K_X,

Classification schemes

- via dimension,
- via degree (in some embedding into projective space); e.g.,

  Fano, general type, intermediate type,

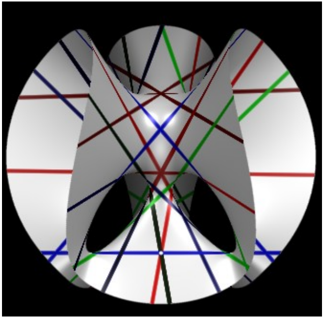
  depending on ampleness of the (anti)canonical class K_X,
- Geometric invariants: rational connectedness, i.e., every pair of points can be connected by a rational curve, or other properties of spaces of rational curves on X.

Classification in low dimensions

dim	**Fano**	**Intermediate type**	**General type**
1	$\mathbb{P}^1$	elliptic curves	$C,\ \mathsf{g}(C) \geq 2$
2	$\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1,$ $X_{2,2} \subset \mathbb{P}^4, X_3 \subset \mathbb{P}^3$	$K3$ surfaces : $X_4 \subset \mathbb{P}^3$ abelian surfaces, ...	...
3	~ 120 families	Calabi – Yau varieties	...

Quadric surfaces

Cubic surfaces



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$$\begin{aligned}
x &= -(s+r)t^2 + (s^2+2r^2)t - s^3 + rs^2 - 2r^2s - r^3 \\
y &= t^3 - (s+r)t^2 + (s^2+2r^2)t + rs^2 - 2r^2s + r^3 \\
z &= -t^3 + (s+r)t^2 - (s^2+2r^2)t + 2rs^2 - r^2s + 2r^3 \\
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What about $x^3+y^3+z^3+2w^3=0$?

Obstruction to rationality is nontriviality of Galois cohomology:

$$H^1(\mathbb{Q}, \mathrm{Pic}(\bar{X})).$$

Del Pezzo surfaces over nonclosed fields

Del Pezzo surfaces = 2-dimensional Fano varieties

Theorem

Let X be a smooth del Pezzo surface.

- $\deg(X) \geq 5$*: If $X(k) \neq \emptyset$ then X is k-rational.*
- $\deg(X) = 4, 3$*: If $X(k) \neq \emptyset$ then X is k-unirational.*

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If $\deg(X) = 1$ then $X(k) \neq \emptyset$. It is unknown whether or not X is unirational. We don't know whether or not rational points are Zariski dense.

Example

Let X be a conic bundle over $\mathbb{P}^1$, over a field k, given by

$$x^2 - ay^2 = f(s)z^2, \quad \deg(f) = 3, \quad \mathrm{disc}(f) = a,$$

with f irreducible over k. Then X is nonrational but stably rational.

Threefolds

The Minimal Model Program implies that rationally connected 3-folds are of three types:

- Fano 3-folds
- Del Pezzo fibrations over $\mathbb{P}^1$
- Conic bundles over a rational surface.

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Many (all??) of these are unirational.

Lüroth's problem

Does unirationality imply rationality?

There were numerous unsuccessful attempts to find counterexamples.

Osservazioni sopra alcune varietà non razionali aventi tutti i generi nulli.

di GINO FANO.

In un lavoro pubblicato alcuni anni or sono negli " Atti " di questa R. Accademia (1) ho dimostrato che la varietà del 4° ordine dello spazio S_4 priva di punti doppi, e la varietà M_3^6 di S_5 intersezione generale di una quadrica e di una varietà cubica di quest'ultimo spazio, pur avendo tutti i generi nulli, non sono razionali. La dimostrazione era fondata sull'impossibilità di soddisfare in pari tempo a certe condizioni, tutte necessarie per l'esistenza di sistemi omaloidici di superficie contenuti rispett. in quelle due varietà.

Counterexamples to Lüroth's problem

Major developments in 1971-72:

- Iskovskikh-Manin: quartic in $\mathbb{P}^4$ via birational rigidity

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- Clemens-Griffiths: cubic in $\mathbb{P}^4$ via intermediate Jacobians
- Artin-Mumford: conic bundles via unramified cohomology

This approach stimulated major developments in algebraic geometry.

Birational rigidity

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- Reid, Corti, Pukhlikov, Cheltsov

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- Reid, Corti, Pukhlikov, Cheltsov
- A smooth hypersurface of degree n in $\mathbb{P}^n$ is birationally rigid (de Fernex, 2013)

Intermediate Jacobians

Theorem

If the intermediate Jacobian $\mathrm{IJ}(X)$ of a threefold X is not a product of Jacobians of curves then X is nonrational.

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Implementation:

- Cubic threefolds (Clemens–Griffiths)
- Intersection of 3 quadrics and conic bundles (Beauville)
- Del Pezzo surface fibrations over $\mathbb{P}^1$ (Alexeev, Vassil Kanev)

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Limitation: Does not detect failure of stable rationality

Idea (Clemens 1974): Let

$$\phi : \mathcal{X} \to B$$

be a family of Fano threefolds, with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X has at most rational double points

(O) Obstruction: the intermediate Jacobian $\mathrm{IJ}(\tilde{\mathcal{X}}_0)$ (of the resolution of singularities $\tilde{\mathcal{X}}_0$) is not a product of Jacobians of curves.

Then a general fiber $\mathcal{X}_b$ is not rational.

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Implementation (Beauville 1977): nonrationality of certain Fano varieties

Brauer group

Theorem (Artin-Mumford)

Let $X \to S$ be a conic bundle over a smooth projective rational surface with discriminant a smooth curve

$$D = \sqcup_{j=1}^{r} D_j \subset S,$$

and with $g(D_j) \geq 1$ for all j. Then

$$H^2_{nr}(k(X), \mathbb{Z}/2) = (\mathbb{Z}/2)^{r-1}.$$

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Implementation: A special conic bundle over $\mathbb{P}^2$.

Cycle-theoretic tools: CH_0

$\mathrm{CH}_0(X_k)$ is the abelian group generated by zero-dimensional subvarieties $x \in X$ (e.g., points $x \in X(k)$), modulo k-rational equivalence.

Assuming $X(k) \neq \emptyset$, there is a surjective homomorphism

$$\deg : \mathrm{CH}_0(X_k) \to \mathbb{Z}.$$

For which X is this an isomorphism?

Example

- X a unirational or rationally-connected variety over $k = \mathbb{C}$.

CH_0-TRIVIALITY

A projective X/k is universally CH_0-trivial if for all k'/k

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For example, smooth k-rational varieties are universally CH_0-trivial. Unirational or rationally-connected varieties are not necessarily universally CH_0-trivial. Varieties with nontrivial unramified cohomology groups are not universally CH_0-trivial.

Specialization method
Voisin 2014, Colliot-Thélène–Pirutka 2015

Let

$$\phi : \mathcal{X} \to B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X has mild singularities

(O) Obstruction: the group $H^2_{nr}(\mathbb{C}(X), \mathbb{Z}/2)$ is nontrivial.

Then a very general fiber of ϕ is not stably rational.

Specialization method: First applications

Very general varieties below are not stably rational:

- Quartic double solids $X \to \mathbb{P}^3$ with ≤ 7 double points (Voisin 2014)
- Quartic threefolds (Colliot-Thélène–Pirutka 2014)
- Sextic double solids $X \to \mathbb{P}^3$ (Beauville 2014)
- Fano hypersurfaces of high degree (Totaro 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of prime degree (Colliot-Thélène–Pirutka 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of arbitrary degree (Okada 2016)

Conic bundles over rational surfaces

Theorem (Hassett-Kresch-T. 2015)

A very general conic bundle $X \to S$, over a rational surface S, with discriminant of sufficiently high degree, e.g., $X \to \mathbb{P}^2$, with discriminant a curve of degree ≥ 6, is not stably rational.

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Theorem (Kresch-T. 2017)

Similar result for 2-dimensional Brauer-Severi bundles over rational surfaces.

Theorem (Hassett-T. 2016)

A very general fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ in quartic del Pezzo surfaces which is not rational and not birational to a cubic threefold is not stably rational.

Del Pezzo fibrations

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Theorem (Krylov-Okada 2017)

A very general del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ of degree 1, 2, or 3 which is not rational and not birational to a cubic threefold is not stably rational.

Fano threefolds

Theorem (Hassett-T. 2016)

A very general nonrational Fano threefold X over $k = \mathbb{C}$ which is not birational to a cubic threefold is not stably rational.

Fano threefolds: idea and implementation

Find suitable degenerations with mild singularities and birational to conic bundles.

Nonrational Fano threefolds with

$$\mathrm{Pic}(V) = -K_V\mathbb{Z} \quad \text{and} \quad d = d(V) = -K_V^3 :$$

- $d = 2$ sextic double solid
- $d = 4$ quartic
- $d = 6$ intersection of a quadric and a cubic
- $d = 8$ intersection of three quadrics
- $d = 10$ section of $\mathrm{Gr}(2,5)$ by two linear forms and a quadric
- $d = 14$ birational to a cubic threefold

Fano threefolds: degenerations

From general quartic del Pezzo $\mathcal{X} \to \mathbb{P}^1$ to Fano threefolds V:

- $d = 2$: $h(\mathcal{X}) = 22 \Rightarrow$ sextic double solid V with 32+4 nodes
- $d = 4$: $h(\mathcal{X}) = 20 \Rightarrow$ quartic threefold with 16 nodes
- $d = 6$: $h(\mathcal{X}) = 18 \Rightarrow$ quadric $\cap$ cubic with 8 nodes
- $d = 8$: $h(\mathcal{X}) = 16 \Rightarrow$ intersection of three quadrics with 4 nodes
- $d = 10$: $h(\mathcal{X}) = 14 \Rightarrow$ specialization of a V with 2 nodes

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The other families of Fano threefolds are conic bundles, but not very general, as in the theorem above. Additional work is needed.

Consider the intersection of two $(1,2)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^4$:

$$sP_1 + tQ_1 = sP_2 + tQ_2 = 0.$$

Let $v_1, \ldots, v_{16} \in \mathbb{P}^4$ denote the solutions to

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- Projection onto the second factor gives a quartic threefold

  $$V := \{P_1 Q_2 - Q_1 P_2 = 0\} \subset \mathbb{P}^4$$

  with 16 nodes $v_1, \ldots, v_{16}$.

Rationality in families: Hassett-Pirutka-T. 2016

There exist smooth families of projective rationally connected fourfolds $\mathcal{X} \to B$ over $k = \mathbb{C}$ such that:

- For every $b \in B$ the fiber X_b is a quadric surface bundle over a rational surface S;
- For very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational;
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Two difficulties:

- Construction of special X satisfying (**O**) and (**S**)

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- Rationality constructions

Rationality in families: implementation

We consider bi-degree $(2,2)$ hypersurfaces

$$X \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

Projection onto the first factor gives a quadric bundle over $\mathbb{P}^2$, its degeneration divisor $D \subset \mathbb{P}^2$ is an octic curve.

Special fiber

Let

$$X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[s:t:u:v]}$$

be a bi-degree $(2,2)$ hypersurface given by

$$yzs^2 + xzt^2 + xyu^2 + F(x,y,z)v^2 = 0,$$

where

$$F(x,y,z) := x^2 + y^2 + z^2 - 2xy - 2yz - 2xz.$$

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The discriminant curve for the projection $X \to \mathbb{P}^2$ is given by

$$x^2y^2z^2F(x,y,z) = 0.$$

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- Desingularization: by hand; the singular locus is a union of 6 conics, intersecting transversally

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- Then the quadric over the function field $\mathbb{C}(\mathbb{P}^2)$ has a point, and X is rational.
- The corresponding locus is dense in the usual topology of the moduli space.

Other applications: Hassett–Pirutka–T. 2017

Let $X \subset \mathbb{P}^7$ be a very general intersection of three quadrics. Then X is not stably rational. Rational X are dense in moduli.

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Idea: Such X admit a fibration $X \to \mathbb{P}^2$, with generic fiber a quadric surface and octic discriminant.

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- dim $= 2$ - rational
- dim $= 3$ - nonrational, are there any stably rational examples?
- dim $= 4$ - periodicity??
- dim $= 7, 8$ - some results by Atanas Iliev and Laurent Manivel

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Unirational parametrizations:

- all admit unirational parametrizations of degree 2
- (Hassett-T. 2001) Cubic fourfolds with an odd degree unirational parametrization are dense in moduli

Addington–Hassett–T.–Várilly-Alvarado 2016

The locus of rational cubic fourfolds in $\mathcal{C}_{18}$ – special cubic fourfolds of discriminant 18 – is dense.

Special cubic fourfolds

Addington–Hassett–T.–Várilly-Alvarado 2016

The locus of rational cubic fourfolds in $\mathcal{C}_{18}$ – special cubic fourfolds of discriminant 18 – is dense.

Idea: Every $X \in \mathcal{C}_{18}$ admits a fibration $X \to \mathbb{P}^2$ with general fiber a degree 6 Del Pezzo surface. A multisection of degree coprime to 3 forces rationality. The locus of such cubics is dense in $\mathcal{C}_{18}$.

Remark

Something like this should work for 6-dimensional cubics.

New approaches to rationality

- Derived categories (Kuznetsov, ...)
- Sheaves of categories, moduli spaces of Landau-Ginzburg models (Katzarkov, ...)

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Summary

- The specialization method of Voisin, further developed by Colliot-Thélène–Pirutka, has triggered new advances in the study of rationality properties of higher-dimensional varieties.
- Stable rationality of general threefolds is essentially settled.
- Rationality properties can change in smooth families in dimension ≥ 4.
- Rationality and stable rationality of cubics remain a challenge.