Boundary behavior of invariant metrics in complex analysis

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The Carathéodory pseudodistance and metric

In 1927, C. Carathéodory defined a pseudodistance on any domain D in $\mathbb{C}^n$ via a "generalized" Schwarz-Pick lemma in order to decide, at least in principle, whether two given domains are not biholomorphic:

$$c_D(z,w) = \sup\{p(f(z), f(w)) : f \in \mathcal{O}(D, \Delta)\},$$

where p is the Poincaré distance of the unit disc Δ.

A specific property of c_D is that holomorphic mappings act as contractions. In particular, biholomorphic mappings operate as isometries, and monotonicity under inclusion of sets take a place: $D \subset G \Rightarrow c_D \geq c_G$. Moreover, c_D is smallest contractible pseudodistance such that $c_\Delta = p$.

The Carathéodory(-Reiffen pseudo)metric is the infinitesimal version of c_D :

$$\gamma_D(z; X) = \sup\{|f'(z)X)| : f \in \mathcal{O}(D, \Delta)\}, \quad z \in D,\ X \in \mathbb{C}^n.$$

The Kobayashi pseudodistance and metric

In 1967, S. Kobayashi defined the largest contractible pseudo-distance, k_D, with the above properties. In fact, k_D is the largest pseudodistance which does not exceed the Lempert function:

$$l_D(z, w) = \inf\{p(\lambda, \mu)) : f \in \mathcal{O}(\Delta, D), f(\lambda) = z, f(\mu) = \mu\}.$$

(Due to J. Globevnik (1976), any two points on a complex manifold can be joined by an analytic disc.)

It turns out (H. Royden, 1971) that k_D is the integrated form of its infinitesimal version, the Kobayashi(-Royden pseudo)metric:

$$\kappa_D(z; X) = \inf\{|\alpha| : f \in \mathcal{O}(\Delta, D) : f(\lambda) = z, \alpha f'(\lambda) = X\}.$$

$$k_D(z, w) = \inf \int_0^1 k_D(\gamma(t); \gamma'(t))dt,$$

where the infimum is taken on all piecewise $\mathcal{C}^1$-smooth curves $\gamma : [0, 1] \to D$ with $\gamma(0) = z$ and $\gamma(1) = w$.

The Bergman kernel, metric and distance

The Bergman kernel K_D of a domain D in $\mathbb{C}^n$ is the reproducing kernel of the Hilbert space $A^2(D)$ of square-integrable holomorphic functions on D (S. Bergman, 1933).

The Bergman kernel on the diagonal solves an extremal problem:

$$K_D(z) := K_D(z,z) = \sup\{|f(z)|^2 : f \in A^2(D), ||f||_D \leq 1\}$$

The Bergman metric β_D is defined as follows:

$$\beta_D(z;X) = \sqrt{\sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K_D(z) X_j \overline{X_k}}.$$

The Bergman metric also solves an extremal problem:

$$\beta_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}}, \quad \text{where}$$

$$M_D(z;X) = \sup\{|f'(z)X| : f \in A^2(D), ||f||_D \leq 1, f(z) = 0\}.$$

The Bergman distance b_D is the integrated form of the Bergman metric.

The Bergman metric and distance are not monotone under inclusion even of plane sets.

The following inequalities hold:

(by the Schwarz-Pick lemma) $\gamma_D \leq \kappa_D \Rightarrow c_D \leq c_D^i \leq k_D \leq l_D$;

(K.T. Hahn, 1976) $\gamma_D \leq \beta_D \Rightarrow c_D^i \leq b_D$.

The Bergman and Kobayashi metrics (distances) are not comparable, in general.

(Strong) pseudoconvexity

A $\mathcal{C}^2$-smooth point $a \in \partial D$ is called Levi pseudoconvex if

$$\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(a) X_j = 0 \Rightarrow \mathcal{L}(r_D(a); X) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(a) X_j \overline{X_k} \geq 0,$$

where r_D is the signed distance to ∂D.

If the last inequality is strict for $X \neq 0$, the point is called strongly pseudoconvex ($\Leftrightarrow$ D is locally biholomorphic (near a) to strong convex domain).

A domain in $\mathbb{C}^n$ is said to be pseudoconvex if it admits an exhaustion plurisubharmonic function.

Pseudoconvexity of a $\mathcal{C}^2$-smooth domain in $\mathbb{C}^n$ is equivalent to Levi pseudoconvexity of all boundary points.

Two applications of the boundary behavior of holomorphic invariants

Theorem M-K-P. (G.A. Margulis, 1971, G.M. Khenkin, 1973, S.I. Pinchuk, 1975) If D and G are C^2-smooth pseudoconvex domains in $\mathbb{C}^n$, and G is strongly pseudoconvex, then any proper holomorphic map $f : D \to G$ admits a 1/2-Hölder extension on $\bar{D}$.

Theorem W-R. (B. Wong, 1977; J.P. Rosay, 1979) Any strongly pseudoconvex domain in $\mathbb{C}^n$ with non-compact group of holomorphic automorphisms is biholomorphic to a ball.

Behavior of the Bergman kernel near strongly psc points

Theorem H. (L. Hörmander, 1965) Let a be a strongly pseudoconvex boundary point of a bounded pseudoconvex domain D in $\mathbb{C}^n$. Then

$$\lim_{z \to a} K_D(z) d_D^{n+1}(z) = \frac{n!}{4\pi^n} p_D(a),$$

where $d_D(z) = \text{dist}(z; \partial D)$ and $p_D(a)$ is the product of the n-1 eigenvalues of the Levi form at a.

This result was improved by C. Fefferman (1974) in order to show that any biholomorphism between $\mathcal{C}^\infty$-smooth strongly pseudoconvex domains extends to a diffeomorphism between their closures.

The Fefferman asymptotic expansion formula for the Bergman kernel on the diagonal: .

Theorem F. (C. Fefferman, 1974) Let D be a $\mathcal{C}^\infty$-smooth, bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. Then, for $z \in D$ near ∂D, one has that

$$K_D(z) = \frac{c_1(z)}{d_D^{n+1}(z)} + c_2(z). \log d_D(z),$$

where c_1 and c_2 belong to $\mathcal{C}^\infty(\bar{D})$, and $c_1|_{\partial D} \neq 0$.

Conjecture R. (I.P. Ramadanov, 1982) If $c_2 = 0$ near ∂D, then D is biholomorphic to a ball.

For $n = 2$, it was proved by D. Burns and R. Graham, and, independently, by L. Boutet de Monvel in 1987.

Behavior of invariant metrics near strongly psc points

Set $s_D(z;X)=\sqrt{\frac{||X_n||^2}{4d_D^2(z)}+\frac{\mathcal{L}(r(z);X_t)}{d_D(z)}}$, where X_n and X_t are the normal and tangent components of X at a point $\pi_D(z)\in\partial D$ such that $||z-\pi_D(z)||=d_D(z)$.

Theorems D&Fu. (K. Diederich, 1970; I. Graham, 1975, S. Fu, 1995) Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. Then

$$\lim_{z\to\partial D}\sup_{X\neq 0}\frac{\beta_D(z;X)}{s_D(z;X)}=\sqrt{n+1},$$

$$\lim_{z\to\partial D}\sup_{X\neq 0}\frac{\gamma_D(z;X)}{s_D(z;X)}=1.$$

Theorem M. (I. Graham, 1975, D. Ma, 1992) Let a be a strongly pseudoconvex boundary point of a domain D in $\mathbb{C}^n$. Then

$$\lim_{z\to\partial D}\sup_{X\neq 0}\frac{\kappa_D(z;X)}{s_D(z;X)}=1.$$

Theorem Fu easily follows by:

• Theorem M;

• Lempert theorem (1981), implying that $\gamma_D = \kappa_D$ on any convex domain D in $\mathbb{C}^n$;

• Fornaess embedding theorem (1977), providing a holomorphic map Φ from a neighborhood of $\bar{D}$ to $\mathbb{C}^n$, and a strongly convex domain $G \supset \Phi(D)$ such that, near a, Φ is 1-1, and $\partial G = \partial \Phi(D)$.

Theorems D and M can be proved, for example, by following the two steps below.

Localization of the Bergman kernel and metric

The main points in the proof of Theorem H are:
- approximation of D near a by complex ellipsoids;
- localization of the Bergman kernel.

Theorem L. (L. Hörmander, 1965) Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let U be a neighborhood of a local holomorphic peak point $a \in \partial D$ (i.e., $\exists p \in \mathcal{O}(D \cap U_a)$: $\lim_{z\to a} |p(z)| = 1 > \sup\{|p| : D \cap U_a \setminus V_a\}$). Then

$$\lim_{z\to a} \frac{K_{D\cap U}(z)}{K_D(z)} = 1.$$

The proof of Theorem L is based on the following L^2 estimate for the $\bar{\partial}$-problem.

Theorem E. (L. Hörmander, 1965) Let D be a pseudoconvex domain in $\mathbb{C}^n$, let $\varphi \in PSH(D)$ and let $e^{\psi} \in C(D)$ be a lower bound for the plurisubharmonicity of φ. For any form $f \in L^2_{(p,q+1)}(D, loc)$ such that $\bar{\partial} f = 0$ and

$$M = \int_D |f|^2 e^{-\varphi-\psi} < \infty$$

one can find a (p, q)-form $u \in L^2_{(p,q)}(D, \varphi)$ such that $\bar{\partial} u = f$ and

$$(q+1) \int_D |u|^2 e^{-\varphi} \leq M.$$

Let p be a local holomorphic peak function for $a \in \partial D$ and let g be an extremal function for $K_{D\cap U}(z)$. Then Theorem L can be proved by solving the $\bar{\partial}$-problem $\bar{\partial} u = \bar{\partial}(\chi p^k g)$ as in Theorem E, choosing in an appropriate way φ (usually, depending on the pluricomplex Green function g_D), a $\mathcal{C}^{\infty}$-smooth cut-off function χ with support in $D \cap U$, and an integer $k = k(z) \to \infty$ as $z \to a$, setting $f = u - \chi p^k g$, and taking $f/||f||_D$ as a competitor for $K_D(z)$.

The proof of Theorem E implies that it remains true if φ is plurisubharmonic on the non-zero set of f.

This observation allows to show that Theorems L holds for any (not necessary bounded) pseudoconvex domain (N.,2002).

The same localization result is true for the Bergman metric instead of the Bergman kernel (N., 2002).

It follows that:

- the condition of boundedness of D in Theorem H is superfluous;
- Theorem D remains true for any pseudoconvex domain D which is strongly pseudoconvex at $a \in \partial D$.

The $\mathcal{C}^{2,\varepsilon}$-smooth case

One may prove a stronger localization near a strongly pseudoconvex boundary point by using a special local holomorphic peak function.

Theorem L1. (N., 2014) Let U be a neighborhood of a strongly pseudoconvex boundary point a of a pseudoconvex domain D in $\mathbb{C}^n$. Then there exist $c > 0$ and a neighborhood $V \subset U$ of a such that for $z \in D \cap V$ and $X \neq 0$, one has that

$$1 < \frac{K_{D\cap U}(z)}{K_D(z)} < 1 - cd_D(z)\log d_D(z),$$

$$1 < \frac{M_{D\cap U}(z;X)}{M_D(z;X)} < 1 - cd_D(z)\log d_D(z).$$

This theorem is inspired by a similar result for the Kobayashi metric due to J.P. Rosay and F. Forstneric, 1987 (without the assumption of pseudoconvexity and the log term).

Theorem L1 allows to refine Theorem D under higher regularity assumptions.

Theorem D1. (N., 2014) Let $\varepsilon \in (0,1]$. Let a be $C^{2,\varepsilon}$-smooth strongly pseudoconvex boundary point of a bounded pseudoconvex domain D in $\mathbb{C}^n$. Then, for $z \in D$ near a, one has that

$$\sup_{X \neq 0} \left| \frac{\beta_D(z;X)}{s_D(z;X)} - \sqrt{n+1} \right| = O(d_D^{\varepsilon/2}(z)).$$

One may formulate similar results for κ_D and γ_D. The $\mathcal{C}^3$-smooth case was considered by D. Ma (1992).

In this case, Theorem D1 was independently proved by K. Diederich and J.E. Fornaess (2015) by using the boundary behavior of the so-called squeezing function.

Roughly speaking, this function, say $s_D(z)$, describes how close is D (up to biholomorphism) to a ball centered at $z \in D$.

The Bergman distance and Gromov hyperbolicity

Theorem D1, combining with a similar result for κ_D, allows to show that $\sup_{D\times D} |b_D - \sqrt{n+1}.k_D| < \infty$ on any $C^{2,\varepsilon}$-smooth bounded strongly pseudoconvex domain D in $\mathbb{C}^n$ (N., 2014).

In particular, b_D and $\sqrt{n+1}.k_D$ are quasi-isometric.

By a result of Z. Balogh and M. Bonk (2000), (D, k_D) is a Gromov hyperbolic space (i.e., $\exists\ \delta > 0$ such that any geodesic triangle is δ-thin).

It follows that (D, b_D) is a Gromov hyperbolic space, too, which gives an affirmative answer of their question.