# Categories and Filtrations

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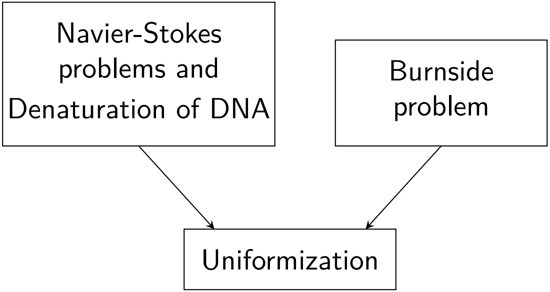
July 10, 2017

# Overview


Navier-Stokes problems and Denaturation of DNA
Burnside problem
Uniformization

# Navier-Stokes

For $V$ - velocity, $P$ - pressure:

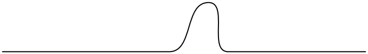
$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V + \frac{1}{P} \nabla P = \nu \Delta V + f(x)$$

$$\nabla V = 0$$

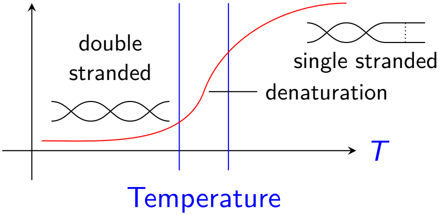
$$\Downarrow \text{L. D. Landau}$$

$$\frac{dA}{dt} = a\mu + b|A|^2 + h.o.t.$$

This produces traveling waves.



# Denaturation of DNA




This is described by a one-dimensional lattice and the following lattice ODE.

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1})$$

$W$ - on-site potential

$V$ - interaction potential

## PDE

Method of Central Manifolds

$$\Downarrow$$

## ODE

$$\frac{1}{\mathcal{J}^2}\frac{d^2y}{dx^2} = V'(y(x+1)-y(x)) - V'(y(x)-y(x-1))$$

$$x = n - \frac{t}{2}$$

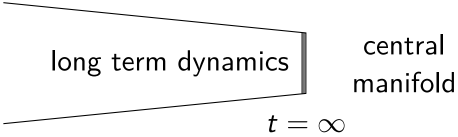
$$u_n(x) = y$$

Recall the method of central manifolds.

$$\begin{array}{c}\text{Central manifolds}\\ \text{of equilibrium points}\end{array} = \left\{ \textit{orbits} \,\middle|\, \begin{array}{c}\text{neither attraction of stable manifold}\\ \text{nor repulsion of unstable manifold}\end{array} \right\}$$

Central manifold is given by the linearization of $\mid$ eigenvalues $\lambda_i$ with $\mathrm{Re}\lambda_i = 0$ or $\lambda_i = 0 \mid$.

$\lambda_i = 0 \Leftrightarrow$ slow manifold spanned by eigenvectors

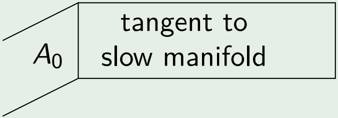



## Recall

If we have a dynamical system

$$\frac{dx}{dt} = f(x) \xrightarrow{\text{linearization}} \frac{dx}{dt} = Ax$$

$A_0$ - eigenvectors with $\lambda = 0$



# Burnside theory

Recall that Burnside group is defined as

$$B(n, d) = \{F^n | x^d = 1, \forall x \in F^n\}$$

$F^n$ is a free group with $n$ generators.

$B(n, 2) = \mathbb{Z}_2^n$
$B(n, 3) =$ finite with order $3^C$, where $C$ depends on the nilpotency class
$B(n, 4) =$ finite (Sanov)
$B(n, 5) =$ ? $(n \geqslant 2)$
$B(n, 6) =$ finite (M. Hall)

**Question**

Can we find $n \geqslant 2$ and $d$ so that $B(n,d)$ is infinite?

**Theorem (Adian, Novikov)**

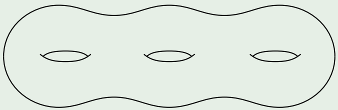
$\exists$ *infinite* $B(n,d)$, *where* $d$ *is odd and* $d \geqslant 4381$.

**Theorem (Olshanski)**

*Let* $\Gamma$ *be a hyperbolic group. Then* $\Gamma(d)$ *is infinite for* $d >> 0$.

## Example

$\Gamma = \pi_1(C)$, $C$ - Riemann surface



## Theorem (Zelmanov)

*$B(n, d)$ is not residually finite if infinite.*

## Question (Zelmanov)

Can we find $n_1 > n_2$ so that $B(n_1, m)$ is infinite and $B(n_2, m)$ is finite?

# Uniformization

Recall:

**Theorem (Riemann)**

*Let $X$ be a one-dim smooth projective variety. Then* $\widetilde{X} = \mathbb{C}, \mathbb{P}^1, \mathbb{D}$.

If $\dim_{\mathbb{C}} X = 2$, then $\widetilde{X} = \mathbb{C} \times \mathbb{C}, \mathbb{P}^2, \mathbb{D} \times \mathbb{D}, \ldots$

**Question (Shafarevich)**

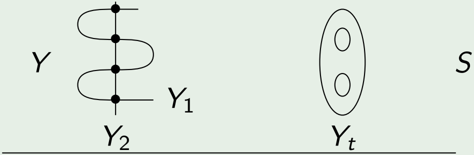
$\widetilde{X}$ is hol. convex for $X$ a smooth projective variety.

Recall:

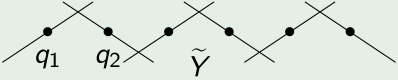
**Definition**

A complex space $M$ is **hol. convex** if $\forall$ sequence of $q_1, \ldots, q_n$ without a limit point, $\exists$ a hol. function unbounded on it.

# Example




$$
\begin{aligned}
\text{Let } \ &\mathrm{Im}(\mathrm{H}_1(Y_1,\mathbb{Z}) \to \mathrm{H}_1(S,\mathbb{Z})) = 0\\
&\mathrm{Im}(\mathrm{H}_1(Y_2,\mathbb{Z}) \to \mathrm{H}_1(S,\mathbb{Z})) = 0\\
\text{and } \ &\mathrm{Im}(\mathrm{H}_1(Y,\mathbb{Z}) \to \mathrm{H}_1(S,\mathbb{Z})) = \infty
\end{aligned}
$$




$\Rightarrow \widetilde{S}$ is not holomorphically convex.

But strictness of MHS implies

$$\mathrm{Im}(\mathrm{H}_1(Y_1,\mathbb{Z}) \to \mathrm{H}_1(S,\mathbb{Z})) = 0$$
$$\mathrm{Im}(\mathrm{H}_1(Y_2,\mathbb{Z}) \to \mathrm{H}_1(S,\mathbb{Z})) = 0$$
$$\Downarrow$$
$$\text{and } \mathrm{Im}(\mathrm{H}_1(Y,\mathbb{Z}) \to \mathrm{H}_1(S,\mathbb{Z})) = 0$$

$$\mathrm{H}_1(S) \to \pi_1^{\mathrm{Malcer}}(S)$$

**Theorem (K)**

*Let $S$ be a smooth projective variety and $\pi_1(S)$ nilpotent. Then $\widetilde{S}$ is holomorphically convex.*

$$\mathrm{H}_1(S, \mathbb{Z}) \to \pi_1^{\mathrm{Malcer}}(S)$$

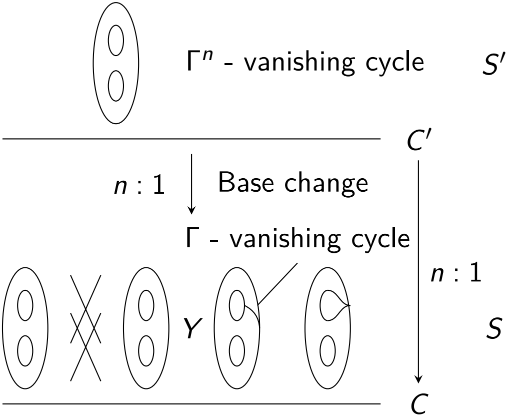
**Theorem (EKPR)**

*Let $X$ be a smooth projective variety and $\pi_1(X) \subset \mathrm{GL}(n, \mathbb{C})$. Then $\widetilde{X}$ is holomorphically convex.*

**Remark**

This technique could lead to $\pi_1(X)$ residually finite.

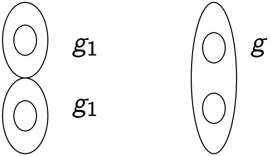
# Uniformization Burnside problem




$$1 \to \pi_1(Y)/_{<\Gamma>} \to \pi_1(S') \to \pi_1(C) \to 1$$

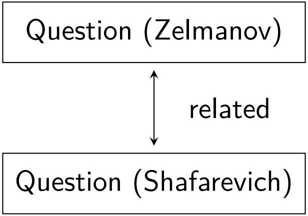
## Theorem (Zelmanov)

*$\pi_1(S')$ is not residually finite.*




$g_1 < g$

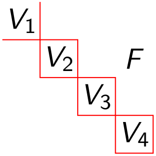
If $\Gamma_{g_1}(m)$ is finite but $\Gamma_g(m)$ is infinite.
$\Rightarrow \widetilde{S}$ is not holomorphically convex.


Question (Zelmanov)
related
Question (Shafarevich)

# Uniformization and Central Manifolds

"Hodge" theory of central manifolds

$\rightarrow$ Vector or Higgs bundles




## Example

Solutions of

$$\frac{\partial H}{\partial t} = \Lambda F + x$$

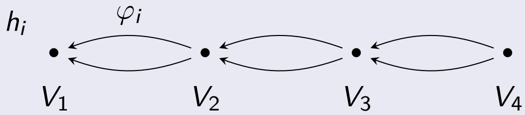
$$\Downarrow \text{ center manifold}$$

$$ODE$$

In fact:

## Theorem (Haiden, Katzarkov, Kontsevich,Pandit)

*This ODE is connected with a quiver:*




$$\mathcal{J}_i h_i^{-1}\frac{dh_i}{dt} = \sum_{\alpha:i\to j} h_i^{-1}\varphi_\alpha^+ h_j \varphi_\alpha - \\ - \sum_{\alpha:i\to j} \varphi_\alpha h_j^{-1}\varphi_\alpha^+ h_i$$

## Theorem (HKKP)

*The asymptotics of ODE define a filtration on the central manifold $Z$.*

The dimension of $Z$ is from $d = 1 + \cdots + 1$ to $k_1^2 + \cdots + k_d^2$.
The asymptotics are

$$\mathbb{R}t + \mathbb{R}\log(t) + \cdots + \mathbb{R}\log(\log(\cdots(t))).$$

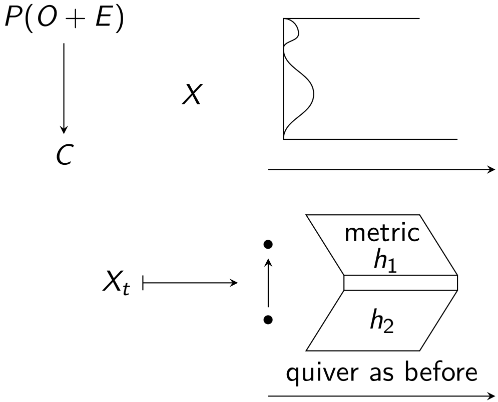
The define a filtration $\mathcal{F}_t$.

## Properties

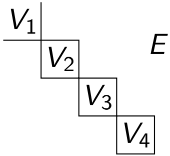
1. $\mathcal{F}$ satisfies Griffiths transversality.
2. $\mathcal{F}$ satisfies functoriality.
3. $\mathcal{F}$ satisfies strictness.
4. The “monodromy” action on $Z$ is semi-simple.

## Theorem (KP)

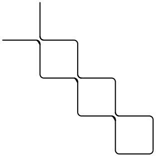
*Monge-Ampère Equation* $\Rightarrow$ *ODE.*

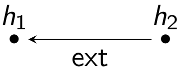


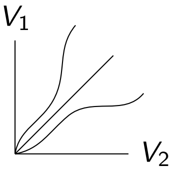
Semistable




Semistable Kähler metric on total space $T(E)$






metrics with very small vectors $V_1 + V_2$

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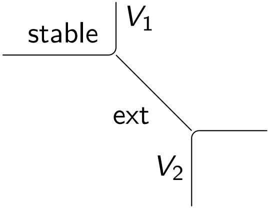
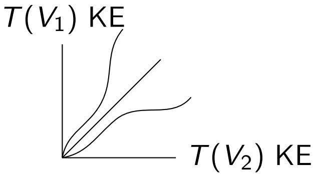
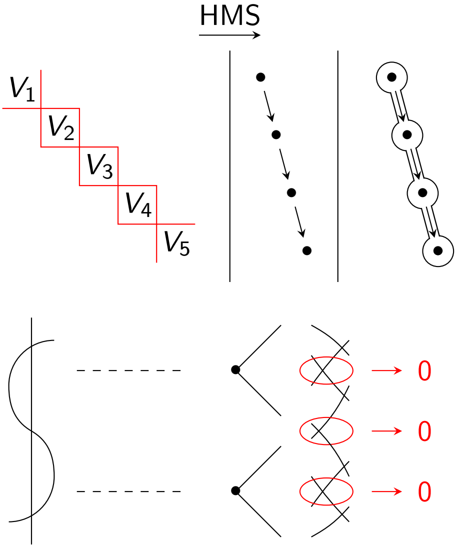
|   |   |
| quiver of semistable metrics<br>$\bullet \longrightarrow \bullet \longrightarrow \bullet$ | quiver of KE metrics<br>$\bullet \longrightarrow \bullet \longrightarrow \bullet$ |

Recall we have a HMS.



$$\pi_1(Y_i) \longrightarrow \mathrm{Fuk}_{FS}(Y_i) = MF$$

$$\Big\downarrow \text{ Mean curvature flow}$$

$\exists$ central manifold with all the properties:

① Functoriality

② Strictness

③ Semi-stability

**Expectation**

①, ②, ③ imply the Shafarevich conjecture.

# Central manifold and Zelmanov conjecture

We have the following:

## Theorem (HKKP)

*Consider* $\mathbb{R}^n$ *as* ${\mathbb{C}^*}^n/{S^1}^n$ *and* $\boxed{\dot{V} = \operatorname{grad} f}$.
*Let* $f$ *be a convex function defined by*

$$f = \sum C_\alpha \cdot < u_\alpha, V > + < u, V >,$$

*where* $u_\alpha, u \in \mathbb{R}^n$.
*Then*

$$V = V_0 t + V_1 \log(t) + \cdots + V_n \log(\log(\cdots(t))) + O(1).$$

**Remark**

For more complicated functions, we get different solutions.

**Theorem (HKKP)**

*In the above situation, we have a central manifold $Z$, with $1 \leqslant \dim Z \leqslant n$.*

**Theorem (HKKP)**

*In the case $G/K$, we have the same for the gradient equation.*

**Conjecture**

The same holds for CAT($W$).

If this is proven, we can directly work with Burnside groups.
If $B(n_1, m)$ does not have central manifold, $B(n_2, m)$ does not have either.
No jumps in cardinality.

We have a sequence (**):

$$PDE \to ODE \to Z \text{ - central manifold}$$

$$\Downarrow$$

$$CATEGORY$$

**Example**

- Donaldson-YM equation
- Mean curvature equation
- Monge-Ampère equation

## Definition

PDE satisfying (**) is called **categorical**.

## Question

Find a sufficient condition so that PDE is categorical.

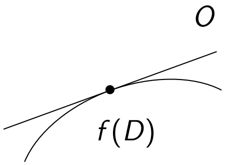
This gives additional structures on solutions e.g. satisfying:

$$\begin{cases} \text{Transversality} \\ \text{Functoriality} \\ \text{Semi-simplicity} \end{cases}$$

Recall: Classical $SL_2$ nilpotent orbit theorem says that for

$$f : D \to U,$$

$\exists$ a nilpotent orbit $O$ s.t.




$O$ approximates $f(D)$.

Consider a spherical functor $S$. Let $\mathrm{D^b}(Q)$ be a quiver category and $\dot{h}$ the flow over metrized objects.

## Conjecture

The superposition of $\dot{h}$ and the flow of $S$ define a new filtration.

## Example 1

$D^b(A_1) = \{V\}$, $N$-PL. This is the standard Hodge filtration.

## Example 2

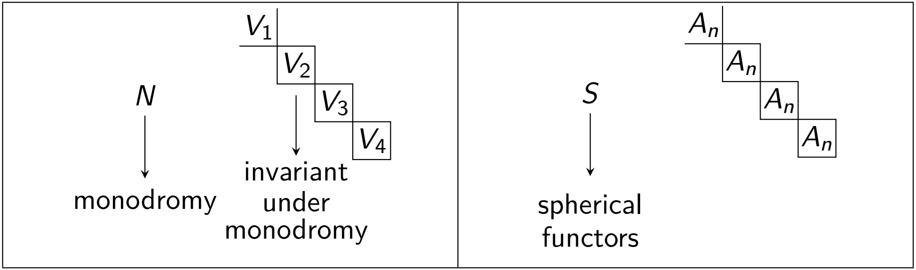
$S$-localization. We get a bifurcation diagram for $\dot{h}$.

## Example 3

Families of matrix factorizations.

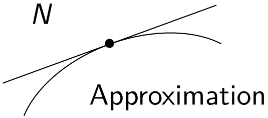
# Multi $SL_2$-Nilpotent Orbit Teorem

We have the following parallel:

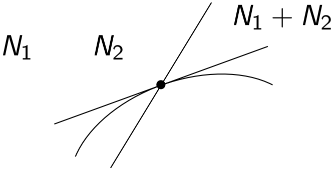


We have the classical $SL_2$-Nilpotent Orbit Theorem:

$$f : D \to U$$



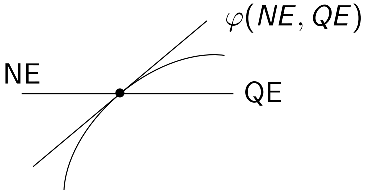

We also have:



We combine two flows:

1. The flow given by Nahm's equation - NE.
2. The "Quiver equation" - QE.

$$\mathcal{J}_i h_i^{-1} \frac{dh_i}{dt} = \sum_{\alpha: i \to j} h_i^{-1} \varphi_\alpha^+ h_j \varphi_\alpha - \sum_{\alpha: i \to j} \varphi_\alpha h_j^{-1} \varphi_\alpha^+ h_i$$




**Conjecture**

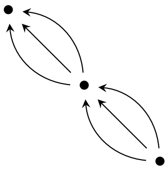
We get a new "refined" filtration.

In terms of stability conditions, we get:

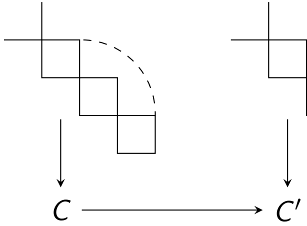
$$\iint_{\Gamma,h} e^h e^{(f)} dzdh \rightsquigarrow \sum_{k,n,m} a_k z^k b_n \log(z) c_m(\underbrace{\log(\log(\cdots}_{m}(h))))$$

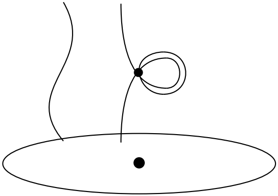
**Conjecture**

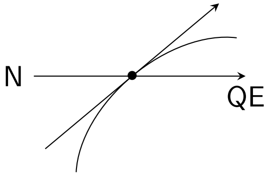
The refined filtration depends on $\operatorname{Ospec} \mathrm{D^b}(\mathcal{C}, A_n)$.



We give a geometric example.






**Proposition**

We have a refined filtration on $\mathrm{D}^{\mathrm{b}}(A_n, \mathcal{C})$.

In general, we have:

(1) A flow $\mathcal{C}$ which defines ODE (associated with a quiver).

(2) Several (spherical) functors $S_1, S_2, \ldots, S_n$.

$$\Downarrow$$

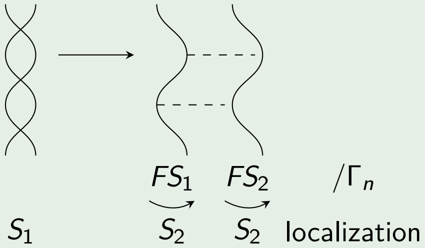
This defines:

A) A refined filtration.

B) A new "Futaki" type of invariant minimizing the refined filtration.

## Example 1

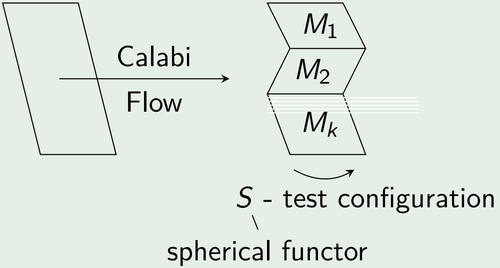
Mean curvature + 2 spherical functors




We get a new filtration with strictness and functoriality. The initial flow is:

$$\dot{c} = -d\,\mathrm{Arg}(\Omega_{|c})\rfloor\omega^{-1}$$

## Example 2

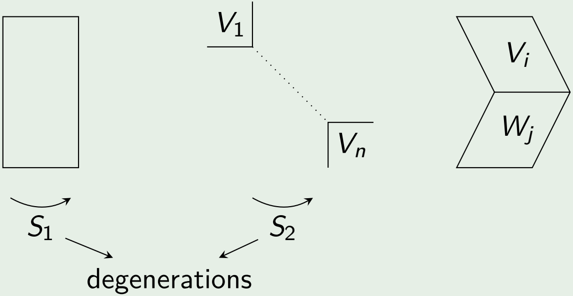



Flow:

$$\dot{\omega} = \omega - \mathrm{Ric}(\omega).$$

Refined Futaki invariant for refined filtration.

## Example 3




Flow: Yang Mills Higgs

- Semi-stable degenerations
- Functoriality
- Strictness

The End