# Automorphisms of Algebras, Multiple Orthogonal Analogs of Classical Orthogonal Polynomials and Bi-orthogonal Ensembles

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**Introduction.** The classical orthogonal polynomials are the most widely used orthogonal polynomials.

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They have a number of properties that distinguish them among the rest of the orthogonal polynomials:

- They are eigenfunctions of an ordinary differential operator
- They have ladder operators, (operators raising or lowering the index)
- They can be presented in terms of hypergeometric functions.
- They can be presented via Rodrigues formulas.
- There are Pearson's equations for the weights of their measures.
- They possess generating functions.

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Hence all the above properties are consequences of the first one.

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Hence all the above properties are consequences of the first one.
They are indispensable tool both in mathematics and physics. E.g. the quantum harmonic oscillator is completely solved in terms of the Hermite polynomials $H_n(x)$, which are eigenfunctions of the operator:

$$L = -\partial_x^2 + x\partial_x$$

Similar properties have the Laguerre and Jacobi orthogonal polynomials, They also have many important applications to mathematics and physics.

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**Important remark.** The orthogonality condition, due to the classical Favard-Shohat theorem is equivalent to the well known 3-term recurrence relation

$$xP_n = P_{n+1} + \beta(n)P_n + \gamma(n)P_{n-1},$$

where $\beta(n), \gamma(n)$ are constants, depending on $n$. Here the polynomials $P_n(x)$, $n = 0, \dots,$ are normalized by the condition that their highest order coefficient is 1.

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where $\beta(n), \gamma(n)$ are constants, depending on $n$. Here the polynomials $P_n(x), \quad n = 0, \ldots,$ are normalized by the condition that their highest order coefficient is 1.
We see that the COP are eigenfunctions of 2 operators - the differential operator in $x$ and the difference operator in $n$.

In recent years more general notions of orthogonality in the space of polynomials were introduced and studied. One of these notions is the notion of multiple orthogonal polynomials (MOP). They have been extensively studied in a number of works by Aptekarev, Nikishin, Sorokin, Van Assche, Kuijlaars, etc.

In recent years more general notions of orthogonality in the space of polynomials were introduced and studied. One of these notions is the notion of multiple orthogonal polynomials (MOP). They have been extensively studied in a number of works by Aptekarev, Nikishin, Sorokin, Van Assche, Kuijlaars, etc.
Applications of MOP.

• Numerical methods. They appear in simultaneous rational approximation of several analytic functions (Hermite-Padé approximation).

• Random matrices.

• Number theory. In fact this is the first application due to Hermite, who used them in his proof of transcendency of "e".

• Spectral theory of nonsymmetric operators

Here I will use a special case of these polynomial systems, called Vector Orthogonal Polynomials (VOP) or $d$-orthogonal polynomials. As in the case of orthogonal polynomials this property has purely algebraic expression, which will be used here. Due to P. Maroni, we can use as a definition of vector orthogonality the following one: the polynomials satisfy a $d+2$-term recursion relation

$$xP_n(x) = P_{n+1} + \sum_{j=0}^{d} \gamma_j(n) P_{n-j}(x)$$

with constants $\gamma_j(n)$, independent of $x$, $\gamma_d \neq 0$. $d = 1$ gives Shohat-Favard theorem.

**Generalized Bochner problem (GBP).** Find systems of polynomials $P_n(x)$, $n = 0, 1, \ldots$ that are eigenfunctions of a differential operator $L$ of order $m$ with eigenvalues $\lambda(n)$ depending on the discrete variable $n$ (the index):

$$LP_n(x) = \lambda(n)P_n(x)$$

*and which at the same time are eigenfunctions of a difference operator, i.e. that satisfy a finite-term (of fixed length $d+2$), recursion relation of the form*

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For each $d$ and $m$ classify all systems of $d$-orthogonal polynomials.

In the case of classical orthogonal polynomials Bochner's theorem shows that all their properties are connected with the differential operator, whose eigenfunctions they are. With the above generalization I hope to find some MOP analogs for the classical orthogonal polynomials. The new MOP's should possess some of their properties.

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**Remark.** Work to find MOP analogs of COP has been successfully done in a different direction. The resulting polynomial systems really have a number of properties that resemble the COP. In particular I have to mention the work by Van Assche with different co-authors - Coussement, Aptekarev, Branquinho. They use of the classical weights and their combinations to construct the weights for the these MOP. However the intersection between my work and these polynomial systems are only the COP.

GBP falls into the framework of the so called **the bispectral problem**. It was isolated by J. J. Duistermaat and F. A. Grünbaum under this name in the influential paper *"Differential equations in the spectral parameter"*, CMP, 1986. The terminology reflects the fact that there is a function $\psi(x, z)$ in two variables, which is an eigenfunction for two operators.

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**Example**

Consider the Meijer's g-functions

$$G^{d,0}_{0,d}\left( {- \atop \alpha_1,\ldots,\alpha_d} \middle| x \right).$$

Put $\psi(x,z) = G^{d,0}_{0,d}\left( {- \atop \alpha_1/d,\ldots,\alpha_d/d} \middle| (-xz/d)^d \right)$. Then $\psi(x,z)$ is an eigenfunction of two differential operators

$$x^{-d}(x\partial_x - \alpha_1)\ldots(x\partial_x - \alpha_d)\psi(x,z) = z^d\psi(x,z)$$
$$z^{-d}(z\partial_z - \alpha_1)\ldots(z\partial_z - \alpha_d)\psi(x,z) = x^d\psi(x,z).$$

This simple example allowed us (Bakalov, Yakimov, H., 1997, CMP) to extend the results of the Duistermaat-Grünbaum paper, which deals with the case $d = 2$.

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The tools that I use here also originate from studies of the bispectral problem. In 1996-1998 my students B. Bakalov and M. Yakimov and myself published a series of papers on the problem. One of them contains a method based on automorphisms of algebras for construction of bispectral operators.

The purpose of the present study is to construct vast families of VOP, and to show that they have many of the properties of the classical orthogonal polynomials. Also connections with other bispectral problems.

**Subalgebras of the Weyl Algebra.** The differential operators with polynomial coefficients form the Weyl algebra $W_1$ which is of major importance for quantum physics. Instead of differential operators in $x$ we can use difference ones and formulate the same problem.

**Subalgebras of the Weyl Algebra.** The differential operators with polynomial coefficients form the Weyl algebra $W_1$ which is of major importance for quantum physics. Instead of differential operators in $x$ we can use difference ones and formulate the same problem.
The construction below can be done in an abstract form (**not an objective in itself!**) and has been done. However I prefer for this talk to use the standard realization.

The Weyl algebra $W_1$ is spanned by the two generators $x, \partial_x$. Denote by $H$ the element $x\partial_x$ and let $R(H)$ be a polynomial of degree $d$ in $H$. Consider also the element $G$ defined as

$$G = R(H)\partial_x \in W_1.$$

We are going to define a subalgebra $\mathcal{B}_1$ of $W_1$, spanned by the generators $x, H, G$.

**Automorphisms**

Define an automorphism $\sigma$ of $\mathcal{B}_1$, acting on elements $A \in \mathcal{B}_1$ as

$$\sigma(A) = e^{\mathrm{ad}_G}(A) = \sum_{j=0}^{\infty} \frac{\mathrm{ad}_G^j A}{j!} \text{ here } \mathrm{ad}_G(A) = GA - AG.$$

## Automorphisms

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### Lemma

*The automorphism $\sigma$ acts locally nilpotently on $\mathcal{B}_1^R$, i.e. the series defining $\sigma(A)$ is finite for any $A$. The images of the generators of $\mathcal{B}_1$ are:*

$$\begin{cases} \sigma(G) = G \\ \sigma(H) = H + G \\ \sigma(x) = x + \sum_{j=0}^{d} \gamma_j(H) G^j. \end{cases} \qquad (1)$$

*with some polynomials $\gamma_j(H)$.*

We can define more general automorphisms by making use of any polynomial $q(G)$ in $G$ and put

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Everything essentially repeats the case of $q(G) = G$.

**Comment.** The above relations and in particular the expressions for $\sigma x$ and $\sigma H$ in fact prepare the corresponding $d + 2$-term recurrence relation and the differential operator. The important point is that we construct both of them simultaneously and together with the VOP.

We need a second algebra $\mathcal{B}_2$, which we define as follows. First

define the algebra $\mathcal{R}$ spanned by the operators $T, T^{-1}, \hat{n}$ acting on functions $f(n)$ as follows

$$T^{\pm}f(n) = f(n \pm 1), \quad \hat{n}f(n) = nf(n).$$

Let us define the map $b\colon W_1 \to \mathcal{R}$ by

$$b(x) = T, \; b(\partial_x) = \hat{n}T^{-1}.$$

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**Corollary**

*The restriction of b is an anti-isomorphism $b\colon \mathcal{B}_1 \to b(\mathcal{B}_1)$ and*

$$b(H) = \hat{n}, \; b(G) = \hat{n} T^{-1} R(\hat{n}).$$

**Definition**

$$\mathcal{B}_2 = b(\mathcal{B}_1)$$

Let us take the simplest polynomial system $\{x^n,\ n = 0, 1, \dots\}$. We see that the above anti-isomorphism $b$ can be realized using the obvious representations of the algebras $\mathcal{B}_1, \mathcal{B}_2$ in $\mathbb{C}$:

$$\begin{aligned} x \cdot x^n &= x^{n+1} \\ Hx^n &= nx^n \\ Gx^n &= nR(n-1)x^{n-1}. \end{aligned}$$

**New anti-isomorphism** Using the automorphism $\sigma$ we can define a new anti-isomorphism $b'$ by the formula $b' = b \circ \sigma^{-1}$. In what follows we will compute explicitly $b'$. The more general $b'_q := b \circ \sigma_q^{-1}$ can also be computed explicitly. Below having in mind applications to VOP we will use the generator $L = \sigma(H)$ instead of $H$.

Also starting with the polynomial system $\psi(x, n) = x^n$ with the help of operator $G$ we define the vector orthogonal polynomials:

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Also starting with the polynomial system $\psi(x, n) = x^n$ with the help of operator $G$ we define the vector orthogonal polynomials:

$$P_n(x) = e^G \psi(x, n) = \sum_{j=0}^{\infty} \frac{G^j \psi(x, n)}{j!}.$$

Notice that the operator $G$ reduces the degree of any polynomial by 1. Hence the above series is finite and defines a polynomial of degree $n$.

Let us introduce the operator $L = \sigma(H)$. In the present situation it reads

$$L = x\partial + R(x\partial)\partial.$$

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**Theorem**

*The polynomials $P_n(x)$ have the following properties:*
*(i) They are eigenfunctions of the differential operator $L$ with eigenvalues $\lambda(n) = n$.*
*(ii) They satisfy the recurrence relation of the form*

$$xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{d} \gamma_j(n) P_{n-j}(x).$$

*(iii) They have ladder (creation and annihilation) operators - $MP_n = P_{n+1}$ and $GP_n = nR(n-1)P_{n-1}$.*

The automorphisms $\sigma_q$ also produce VOP. In particular the cases of $q(G) = \rho G^m$ give a direct generalization of the well known Gould-Hopper polynomials, which correspond to $G = \partial_x$. For this reason we propose to call the polynomials corresponding to $q(G) = \rho G^m$ with any $G$ **Generalized Gould-Hopper Polynomials** (GGHP).

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The GGHP have many features similar to the Hermite polynomials. This will be seen by examples.

## Example

(i) $G = -(x\partial_x + \alpha + 1)\partial_x$ corresponds to Laguerre polynomials.

(ii) $q(G) = \rho\partial_x^m/m$ corresponds to Gould-Hopper polynomials. When $m = 2$ and $\rho = -1$ these are the Hermite polynomials.

(iii) Let again $G = (x\partial_x + \alpha + 1)\partial_x$ and $q(G) = G^2$. This is the simplest new example.

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**Remark.** Earlier Ben Cheikh and Douak defined the VOP, corresponding to $q(G) = G$ by hypergeometric formulas.

**Hypergeometric representation.**

$$
{}_pF_q\left[\begin{matrix}\alpha_1 & \dots & \alpha_p \\ \beta_1 & \dots & \beta_q\end{matrix};x\right]=\sum_{j=0}^{\infty}\frac{(\alpha_1)_j\dots(\alpha_p)_j}{(\beta_1)_j\dots\beta_q)_j}\frac{x^j}{j!}
$$

Here $(a)_j = a(a+1)\dots(a+j-1)$, $(a)_0 = 1$ is the Pochhammer symbol.

**Hypergeometric representation.**

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Here $(a)_j = a(a+1)\dots(a+j-1),\ (a)_0 = 1$ is the Pochhammer symbol.

**Example**

Let $G = R(H)\partial$, where $R(H) = \prod_{k=1}^{d}(H+\alpha_k+1)$. Then

$$ P_n(x) = \prod_{k=1}^{d}(\alpha_k+1)_n \cdot {}_1F_d\left[\begin{matrix} & -n & \\ \alpha_1 & \dots & \alpha_d\end{matrix}; -x\right]. $$

## Example

Generalized Gould-Hopper polynomials have hypergeometric representation. Let $G = (x\partial + \alpha + 1)\partial$, $q(G) = G^2$ and put $n = 2m + i$.

$$P_n(x) = \begin{cases} {}_1F_3\left[\begin{matrix} -m \\ \frac{1}{2} \; \frac{\alpha+1}{2} \; \frac{\alpha+2}{2} \end{matrix}; -(4x)^2\right], & i = 0 \\ x \cdot {}_1F_3\left[\begin{matrix} -m \\ \frac{3}{2} \; \frac{\alpha+3}{2} \; \frac{\alpha+2}{2} \end{matrix}; -(4x)^2\right], & i = 1. \end{cases}$$

Notice the resemblance of this system to Hermite polynomials.

The entire construction works when we represent the Weyl algebra in terms of difference operators instead of differential ones. Also all the properties of the continuous polynomial systems have analogs in this case. Moreover one can present an intertwining operator between the continuous and discrete versions of the algebra $\mathcal{B}_1$, which transforms the corresponding systems into one another.

**Vector measures and Pearson equations.** Using the differential equation for $P_n(x)$ one can derive differential equations and explicit expressions for the measures of VOP. For each polynomial system $\{P_n(x)\}$, $\deg P_n(x) = n$, $n = 0, 1, \dots$ one can define a dual system of linear functionals $v_n$ on $\mathbb{C}[x]$,

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$$\langle v_j, P_n(x) \rangle = \delta_{jn}.$$

We extend the operators $\partial_x$ and multiplication by $f(x)$ to the space of the functionals by:

$$\langle v_j, \partial_x P_n(x) \rangle = \langle -\partial_x v_j, P_n(x) \rangle$$
$$\langle v_j, f(x) P_n(x) \rangle = \langle f(x) v_j, P_n(x) \rangle .$$

Start with the equation

$$LP_n(x) = nP_n(x).$$

It is easy to show that the weight $v_0$ satisfies the equation $L^* v_0 = 0$, where $L^* = -\partial_x[R(-\partial_x x) + x]$.

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$$LP_n(x) = nP_n(x).$$

It is easy to show that the weight $v_0$ satisfies the equation $L^* v_0 = 0$, where $L^* = -\partial_x[R(-\partial_x x) + x]$.
Closer inspection shows that $v_0$ satisfies even simpler equation:

$$[R(-\partial_x x) + x]v_0 = 0.$$

By induction we can prove that the rest of the weights satisfy

$$[R(-\partial_x x + j) + x]v_j = 0,$$

These are the Pearson equations, whose solutions are up to multiplicative constant Meijer's g-functions

$$v_j(x) = G^{d+1,0}_{1,d+1}\left( {}^{-j}_{0,\alpha_1,\ldots,\alpha_d} \middle| x \right)$$

One can easily show that all the weights are linear combinations of $v_0, \ldots, v_{d-1}$ with polynomial coefficients.

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Notice that as mentioned earlier the function

$$v_0(x) = G^{d,0}_{0,d}\left( {- \atop \alpha_1, \ldots, \alpha_d} \,\middle|\, x \right)$$

participates in entirely different bispectral problem. Namely

$$\psi(x,z) = G^{d,0}_{0,d}\left( {- \atop \alpha_1/d, \ldots, \alpha_d/d} \,\middle|\, (-xz/d)^d \right) = v_0((-xz/d)^d)$$

is a joint eigenfunction for two operators.

**Question: Is there a deeper connection between the two bispectral problems or it is just a coincidence?**

The GGHP also have Pearson equations very similar to the above ones. Their solutions are again Meijer's g-functions.

**Example**

Let $G = (x\partial + \alpha + 1)\partial$ and $q = G^2$. Then the weights are given up to multiplicative constant by

$$v_n(x) = G^{3,0}_{1,4}\left( {-n \atop 0, \frac{\alpha+2}{2}, \frac{\alpha+3}{2}, \frac{1}{2}} \,\middle|\, x^2/8 \right)$$

The above constructions can be used to present alternative approach to some well known bi-orthogonal ensembles and eventually to new ones. Among them are the bi-orthogonal ensembles describing products of random matrices studied by Kuijlaars and Zhang.

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Kuijlaars and Zhang considered a bi-orthogonal ensemble defined by the probability density function

$$P(x_1, \dots, x_n) = \prod_{j<k}(x_k - x_j) det[v_{k-1}(x_j)]_{j,k=1,\dots,n},$$

where $v_k(x)$ are the above weights and study the correlation kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} P_n(x) v_n(y)$$

From the weights they compute the polynomials. I construct the polynomials and then find the weights.

### Example

Consider the polynomial system $P_n(x)$ defined by

$$R(H) = \prod_{s=1}^{d}(H + \frac{\alpha + s}{d}),\ \alpha > -1.$$

Toscano introduced the system $Z_n(x) = P_n(x^d/d^d)$. Later Konhauser defined another system $Y_n(x)$, which is bi-orthogonal to $Z_n(x)$. They are simply the weights for the Toscano polynomials modulo a common factor $x^\alpha e^{-x}$. We obtain the Konhauser polynomials directly from the VOP with Bochner's property.

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One can construct the weights for $P_n(x)$ and then make the change of the variables $x = u^d$. Alternatively one can do the computations repeating the above construction for $Z_n(x)$. Then one gets $v_0(x) = x^\alpha e^{-x}$.

Borodin's construction of the kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} P_n(x) v_n(y)$$

for the case of Konhauser polynomials follows easily from or in parallel with the Kuijlaars-Zhang constructions of the kernels.

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The above approach can also be applied to GGH polynomials. By this I mean that I can compute explicitly the weights using the differential operator. However I don't claim that I have analyzed any asymptotics. It is an open problem.
Let us consider the simplest example of the original Gould-Hopper polynomials. They have the following dual system (up to multiplicative constant)

$$v_j(x) = \int_C u^j e^{(-u)^{l+1}/l+xu} \mathrm{d}u.$$

They are linear combinations of $v_0, \ldots, v_{l-1}$ with polynomial coefficients. For $l = 2$ $v_0(x)$ is the Airy function. The function $\psi(x,z) = v_0(x+z)$ is a solutions of another bispectral problem:

$$[\partial_x^l - x]\psi(x,z) = z\psi(x,z), \;\; [\partial_z^l - z]\psi(x,z) = x\psi(x,z).$$

In this connection it was studied by Bakalov, Yakimov, H. in 1997 for all $l \geq 2$. (The case $l = 2$ was studied by DG). We once again come across a weight that is a solution of another bispectral problem.

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**Is it possible to use this fact for more detailed studies of the VOP?**

## Example

$l = 2$ Gould-Hopper polynomials $P_n$; weights $v_0,\ v_1 = v_0'$.

$$(-\partial_x^3 + x\partial_x)P_n = nP_n; \quad v_0 = \int_C \exp(z^3/3 - xz)\mathrm{d}z$$

Airy function.

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Airy function.

## Example

$G = (4H^2 + 4H - \alpha^2 + 1)\partial_x$; weights $v_0$, $v_1 = (xv_0)'$ .

$$(x^2\partial_x^3 + 3x\partial_x^2 - \alpha^2 + x)\partial_x P_n = nP_n$$
$$v_0 = \left(\frac{2}{x}\right)^\alpha \frac{1}{2\pi i}\int_C \exp(x^2 z/4 + z^{-1})\frac{\mathrm{d}z}{z^{\alpha+1}}.$$

Modified Bessel function of the first kind.

Finally I would like to draw your attention to the following observation. The above weights could be considered as stationary wave functions of Gelfand-Dickey hierarchies. Similarly the VOP are wave functions of the bi-graded Toda lattice. What is the connection between these special solutions of the two hierarchies?

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In particular the simplest case $l = 2$ ( Airy function) is closely connected with the Kontsevich matrix model. Notice the integral representation of the weight, which becomes Kontsevich's integral when instead of integrating over the complex number we integrate over $N \times N$ Hermitian matrices.

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The famous Kontsevich theorem shows that in the case $l = 2$ the coefficients of the corresponding solution of KdV have a beautiful combinatorial and algebro-geometric interpretation. Is there such interpretation for the corresponding VOP ?

Other cases like the classical 1-matrix Hermitian model, generalized Kontsevich, and Penner models also have interesting combinatorial interpretations and are present in our scheme. Also the corresponding solutions have a nice representation-theoretic interpretation - they satisfy the so-called "string equation" or Virasoro constraints. In all cases it is crucial that there exists an integral representation. The point is that all the combinatorics stems from the Feynman diagram techniques that is applied to the matrix integrals.

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Is it possible to find such interpretations in other cases?

Can we transform the information to the dual system, i.e. from VOP to the corresponding weights and vice versa?

1. Arno B.J. Kuijlaars, Lun Zhang, Singular values of products of Ginibre random matrices, multiple orthogonal polynomials and hard edge scaling limits, Communications in Mathematical Physics, 332 (2014), 759–781.
2. S. Lando, A. Zvonkin, Graphs on Surfaces and Their Applications, Springer, 2004
3. A. Alexandrov, Cut-and-join description of generalized Brezin-Gross-Witten model, arXiv:1608.01627

Thank you!