The Generalized Kähler Geometry of Holomorphic Symplectic Manifolds

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Plan of the talk

1. The Calabi program and Calabi–Yau manifolds

2. Generalized Kähler geometry

3. Calabi–Yau conjecture in generalized Kähler geometry
Calabi–Yau complex manifolds

Definition (Calabi–Yau manifold)

A compact complex $m$-dimensional manifold $X^m_{\mathbb{C}} = (M^m_{\mathbb{R}}, J)$ is **Calabi–Yau** if
Calabi–Yau complex manifolds

Definition (Calabi–Yau manifold)

A compact complex $m$-dimensional manifold $X^m = (M_{2m}, J)$ is Calabi–Yau if

- $X$ admits a Kähler metric $\omega_0$. 

In a holomorphic chart $\omega_0 = \sqrt{-1} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_j} f := \sqrt{-1} \sum_{i,j=1}^{m} (\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}) (z) \, dz_i \wedge d\bar{z}_j$ with $\left( \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)(z) > 0$, and

$\Theta = \theta(z) \, dz_1 \wedge \cdots \wedge dz_m$ with $\theta(z)$ holomorphic and $\theta(z) \neq 0$. 

Calabi–Yau complex manifolds

Definition (Calabi–Yau manifold)

A compact complex $m$-dimensional manifold $X^m = (M_\mathbb{R}^{2m}, J)$ is **Calabi–Yau** if

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$$\omega_0 = \sqrt{-1} \partial \bar{\partial} f := \sqrt{-1} \sum_{i,j=1}^{m} \left( \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)(z) dz_i \wedge d\bar{z}_j$$

with $\left( \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)(z) > 0$, 

- $K_X = \mathcal{O}_X$, i.e. $X$ admits a nowhere-vanishing holomorphic section $\Theta \in H^0(K_X) = H^0(X, \Omega^m)$. $\Theta = \theta(z) dz_1 \wedge \cdots \wedge dz_m$ with $\theta(z)$ holomorphic and $\theta(z) \neq 0$. 

Calabi–Yau complex manifolds

Definition (Calabi–Yau manifold)

A compact complex $m$-dimensional manifold $X_m = (M_{2m}^2, J)$ is **Calabi–Yau** if

- $X$ admits a Kähler metric $\omega_0$. In a holomorphic chart

$$\omega_0 = \sqrt{-1} \partial \bar{\partial} f := \sqrt{-1} \sum_{i,j=1}^{m} \left( \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)(z) dz_i \wedge d\bar{z}_j$$

with $\left( \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)(z) > 0$, and

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$$\Theta = \theta(z) dz_1 \wedge \cdots \wedge dz_m$$

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Calabi–Yau complex manifolds

Examples of CY manifolds

- (tori) \( X^m = \mathbb{C}^m / (\mathbb{Z}^m \oplus \sqrt{-1}\mathbb{Z}^m) = T^m_{\mathbb{C}} \) with

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\omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^{m} dz_j \wedge d\bar{z}_j, \quad \Theta = dz_1 \wedge \cdots \wedge dz_m.
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Calabi–Yau complex manifolds

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- $X^m \subset \mathbb{P}^{m+1}$ of degree $m + 2$ is CY ($X$ is projective with $K_X = O$).
Calabi–Yau complex manifolds

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• $X^m \subset \mathbb{P}^{m+1}$ of degree $m + 2$ is CY ($X$ is projective with $K_X = \mathcal{O}$).

• Deforming the complex structure in the above examples leads to CY manifolds: each elliptic complex curve and each $K3$ complex surface is CY.
The Kähler geometry of CY manifolds

Definition (Kähler class)

A **Kähler class** of Kähler metrics on $X$ is the space of smooth functions

$$K_{[ω_0]} := \{ ϕ ∈ C^∞(X) : ω_ϕ := ω_0 + \sqrt{-1}∂∂ϕ > 0 \},$$

where $ω_0$ is a given (reference) Kähler metric.
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$[\omega_\varphi] = [\omega_0] = \alpha \in H^2_{dR}(X, \mathbb{R})$. 
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Example (**dim}_\mathbb{C}X = 1**)

Any riemannian metric is determined in a holomorphic chart by $\omega_0 = \sqrt{-1}h(z)dz \wedge d\bar{z}, h > 0$. 


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$$\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \iff \Delta_{\omega_0} \varphi = (1 - e^\psi) \iff \int_X \omega = \int_X \omega_0.$$
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$$K_{[\omega_0]} = \{ \text{volume normalized conformal class on } X \}.$$
The Kähler geometry of CY manifolds

Definition (Ricci form)

The **Ricci form** of a Kähler metric $\omega$ on $X$ is

$$
\rho_\omega := \sqrt{-1} \partial \bar{\partial} \log \left( \frac{\Theta \wedge \bar{\Theta}}{\omega^m} \right)
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where $\Theta$ is any local holomorphic $(m,0)$-form.
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Example ($\dim_{\mathbb{C}} X = 1$)

Any riemannian metric is determined in a holomorphic chart by $\omega = \sqrt{-1} h(z) dz \wedge d\bar{z}$, $h > 0$. For $\Theta = dz$ we get

$$\rho_\omega = K(z) \omega$$

where $K(z) = -\frac{1}{h(z)} \Delta_0 \log h(z)$ is the Gauss curvature.
The Kähler geometry of CY manifolds

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where \( \Theta \) is any local holomorphic \((m,0)\)-form.

**Theorem (Yau)**

*Let \((X^m, \Theta)\) be a CY manifold. Then, \exists unique*

\[
\omega_{CY} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \in K_{\omega_0}
\]

*such that*

\[
\rho_{\omega_{CY}} = 0
\]
The Kähler geometry of CY manifolds

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Theorem (Yau)
Let $(X^m, \Theta)$ be a CY manifold. Then, $\exists$ unique

$$\omega_{\text{CY}} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \in K[\omega_0]$$

such that

$$\rho_{\omega_{\text{CY}}} = 0 \iff (\omega_{\text{CY}})^m = \lambda (\Theta \wedge \bar{\Theta}), \lambda = \text{const.}$$
The Kähler geometry of CY manifolds

Definition (Kähler Ricci flow)

The **Kähler Ricci flow** starting from a Kähler metric \( \omega_0 \) on \( X \) is any smooth family of Kähler metrics \( \omega_t \) solving the geometric PDE

\[
\frac{\partial}{\partial t} \omega_t = -2\rho \omega_t, \quad (\omega_t)|_{t=0} = \omega_0.
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The Kähler geometry of CY manifolds

Definition (Kähler Ricci flow)
The Kähler Ricci flow starting from a Kähler metric $\omega_0$ on $X$ is any smooth family of Kähler metrics $\omega_t$ solving the geometric PDE

$$\frac{\partial}{\partial t} \omega_t = -2\rho \omega_t, \quad (\omega_t)|_{t=0} = \omega_0.$$

Theorem (Cao)
Let $(X^m, \Theta)$ be a CY manifold. Then, for any Kähler metric $\omega_0$ the solution to the Kähler-Ricci flow exists for all $t \in [0, +\infty)$, $\omega_t \in K[\omega_0]$ and $\lim_{t \to \infty} \omega_t = \omega_{CY}$ in $C^\infty$. 
The Kähler geometry of CY manifolds

Summary: The Calabi Program

- The Kähler geometry is described in terms of Kähler classes $K_\alpha$ where $\alpha = [\omega_0] \in H^2_{d\Omega}(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{C})$ runs over the Kähler cone.

- (uniqueness) Each Kähler class $K_\alpha$ contains a unique canonical representative $\omega_{CY,\alpha}$ and any other Kähler metric $\omega \in K_\alpha$ is written

$$\omega = \omega_{CY,\alpha} + \sqrt{-1} \partial \bar{\partial} \varphi, \quad \varphi \in C^\infty(X).$$

- (connectedness) The Kähler Ricci flow allows one to reach the canonical representative $\omega_{CY,\alpha}$. 
Generalized Kähler geometry

(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Definition (GK structure)

A generalized Kähler structure (GK) on a (real) 2m-dimensional manifold $M_{\mathbb{R}}^{2m}$ is defined by the data $(I, J, g, b)$ where:

- $I$ and $J$ are two complex structures on $M_{\mathbb{R}}^{2m}$;
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- $I$ and $J$ are two complex structures on $M^{2m}_\mathbb{R}$;
- $g$ is a Riemannian metric compatible with $I$ and $J$, i.e.

$$g(J\cdot, J\cdot) = g(I\cdot, I\cdot) = g(\cdot, \cdot).$$
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  \[ g(J\cdot, J\cdot) = g(I\cdot, I\cdot) = g(\cdot, \cdot). \]
- $b$ is a 2-form;
- a first order compatibility relation
  \[ \partial_I\omega_I = \sqrt{-1}\bar{\partial}_I(b^2,0), \quad \partial_J\omega_J = -\sqrt{-1}\bar{\partial}_J(b_j^2,0), \]
  
  where $\omega_I = gI$, $\omega_J = gJ$ are the Kähler forms.
Generalized Kähler geometry
(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Example (trivial)

$X = (M^{2m}, I)$ a complex manifold and $(g, \omega_I)$ a Kähler metric. Then, letting $J := -I, b := 0$ we obtain a GK structure $(g, I, J, b)$. 
Generalized Kähler geometry
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Problem

Are there non-trivial examples?
Generalized Kähler geometry
(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Theorem (Gauduchon–Grantcharov–A. for $m = 2$; Hitchin for $m \geq 2$)

Let $(M^{2m}, g, I, J, b)$ be GK and $\sigma := (IJ - JI)g^{-1} \in \Gamma(\wedge^2 TM)$. Then $\sigma_I := \sigma - \sqrt{-1}(I\sigma) \in H^0(M, \wedge^2 (T^1,0 M))$ is Holomorphic Poisson on $X = (M, I)$,
Generalized Kähler geometry

(after Gates–Hull–Rocek, Hitchin, Gualtieri)

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$$\sigma_I = \frac{1}{2} \sum_{i,j=1}^{m} \sigma_{ij}(z) \left( \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \right), \quad (\sigma_{ij} = -\sigma_{ji})$$

with $\sigma_{ij}(z)$ holomorphic, and

$$[\sigma_I, \sigma_I] = 0 \iff \sum_{\ell=1}^{m} \left( \sum_{(ijk) \in S_3} \sigma_{i\ell}(z) \frac{\partial \sigma_{jk}}{\partial z_\ell}(z) \right) = 0.$$
Generalized Kähler geometry

(after Gates–Hull–Rocek, Hitchin, Gualtieri)

**Theorem (Gauduchon–Grantcharov–A. for \( m = 2 \); Hitchin for \( m \geq 2 \))**

Let \((M^{2m}, g, I, J, b)\) be GK and \(\sigma := (IJ - JI)g^{-1} \in \Gamma(\wedge^2 TM)\). Then \(\sigma_I := \sigma - \sqrt{-1}(I\sigma) \in H^0(M, \wedge^2(T^{1,0}_I M))\) is **Holomorphic Poisson** on \(X = (M, I)\), i.e. in a holomorphic chart

\[
\sigma_I = \frac{1}{2} \sum_{i,j=1}^{m} \sigma_{ij}(z) \left( \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \right), \quad (\sigma_{ij} = -\sigma_{ji})
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with \(\sigma_{ij}(z)\) holomorphic, and

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\]

**Corollary (Gualtieri–A.)**

If \(X\) is complex surface of general type not covered by \(\mathbb{D} \times \mathbb{D}\), then

\(\nexists\) non-trivial GK structures.
Generalized Kähler geometry
(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Some basic open problems
Let $X = (M^{2m}, I)$ be a compact complex manifold and $\sigma_I \neq 0$ is a holomorphic Poisson structure.

- Is there a non-trivial GK structure $(g, I, J, b)$ with
  $\sigma = (IJ - JI)g^{-1} = \text{Re}(\sigma_I)$?
  True if $m = 2$ (Goto) or if $(X, \sigma_I)$ is a toric variety (Boulanger).

- If $(X, \sigma_I)$ admits a compatible GK structure does $X = (M, I)$ admit a Kähler metric?
  True if $m = 2$ (Gauduchon–Grantcharov–A., Gualtieri–A.)

- Describe the GK geometry of $(X, \sigma_I)$ in a similar way as we described the Kähler geometry of a CY manifold.
Non-degenerate GK structures

Definition (Non-degenerate GK structure)

The GK structure \((g, I, J, b)\) on \(M^{2m}\) is called **non-degenerate** if the holomorphic Poisson structure

\[
\sigma_I = \sigma - \sqrt{-1}(I\sigma) = \frac{1}{2} \sum_{i,j=1}^{m} \sigma_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}
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is non-degenerate, i.e. \(\det_{\mathbb{C}}(\sigma_{ij}(z)) \neq 0\)
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is non-degenerate, i.e. \(\det_{\mathbb{C}}(\sigma_{ij}(z)) \neq 0\) \(\iff m = 2n\)

\[\iff \sigma_I : T^*_X \cong T_X\text{ where } X = (M, I);\]
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- \(\iff \sigma_I : T_X^* \cong T_X \) where \(X = (M, I)\);
- \(\iff \sigma : T_M^* \cong T_M \).
Non-degenerate GK structures

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$$

is non-degenerate, i.e. $\det_{\mathbb{C}}(\sigma_{ij}(z)) \neq 0 \iff m = 2n$

- $\Leftrightarrow \sigma_I : T^*_X \cong T_X$ where $X = (M, I)$;
- $\Leftrightarrow \sigma : T^*_M \cong T_M$;
- $\Leftrightarrow \sigma_J : T^*_Y \cong T_Y$ where $Y = (M, J)$. 
Non-degenerate GK structures

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The GK structure \((g, I, J, b)\) on \(M^{2m}\) is called **non-degenerate** if the holomorphic Poisson structure

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\]

is non-degenerate, i.e. \(\text{det}_{\mathbb{C}}(\sigma_{ij}(z)) \neq 0\)

- \(\iff \Omega_I = \sigma_I^{-1}\) is closed and non-degenerate \((2, 0)\)-form on \(X\),
- \(\iff \Omega = \sigma^{-1} = \text{Re}(\Omega_I)\) a closed and non-degenerate real \(2\)-form on \(M\),
- \(\iff \Omega_J = \sigma_J^{-1}\) is closed and non-degenerate \((2, 0)\)-form on \(Y\).
Non-degenerate GK structures: revisited

Lemma (Reduction of non-degenerate GK structures)

On $M^{4n}$ we have a bijection

$$\{\text{non-degenerate GK structures}\} \longleftrightarrow \{\(\Omega_I, \Omega_J\)\}$$

where $\Omega_I, \Omega_J$ are closed complex-valued 2-forms satisfying

1. $\text{Re}(\Omega_I) = \text{Re}(\Omega_J) = \Omega$ is a real symplectic form;
2. $\text{Im}(\Omega_I) = \Omega \circ I$, $\text{Im}(\Omega_J) = \Omega \circ J$ for $I, J$ integrable almost complex structures;
3. $\omega_I := -2\left(\text{Im}(\Omega_J)\right)^{1,1}_I > 0 \ (g = -\omega_I \circ I)$. 


Holomorphic symplectic manifolds

Definition (holomorphic symplectic manifold)

A **holomorphic symplectic** manifold is a smooth, compact, complex \( m = 2n \) dimensional manifold \( X^{2n} = (M^{4n}, I) \) which admits a closed non-degenerate \((2, 0)\)-form \( \Omega_I \).
Definition (holomorphic symplectic manifold)

A **holomorphic symplectic** manifold is a smooth, compact, complex $m = 2n$ dimensional manifold $X^{2n} = (M^{4n}, I)$ which admits a closed non-degenerate $(2, 0)$-form $\Omega_I$.

In a holomorphic chart:

$$\Omega_I = \frac{1}{2} \sum_{i,j=1}^{2n} \omega_{ij}(z) dz_i \wedge dz_j, \quad (\omega_{ij} = -\omega_{ji})$$

with $\omega_{ij}(z)$ holomorphic functions, s.t. $\det_{\mathbb{C}}(\omega_{ij}(z)) \neq 0$ and

$$\sum_{(ijk) \in S_3} \frac{\partial \omega_{ij}}{\partial z_k}(z) = 0.$$
Holomorphic symplectic manifolds

Definition (holomorphic symplectic manifold)
A \textbf{holomorphic symplectic} manifold is a smooth, compact, complex $m = 2n$ dimensional manifold $X^{2n} = (M^{4n}, I)$ which admits a closed non-degenerate $(2, 0)$-form $\Omega_I$.

Fact
If $(X, \Omega_I)$ is holomorphic symplectic then

\[
\Theta := (\Omega_I)^n = \left( \det_{\mathbb{C}}(\omega_{ij}(z)) \right)^{\frac{1}{2}} dz_1 \wedge \cdots \wedge dz_{2n}
\]

trivializes $K_X$, i.e. $X$ is CY if it admits a Kähler metric.
Non-degenerate GK structures: Examples

Suppose $X = (M, I)$ is holomorphic symplectic and CY.

Fact (Bogomolov)

Any Calabi–Yau metric $g_{\text{CY}}$ on $X$ is hyper-Kähler, i.e. $g_{\text{CY}}$ is Kähler with respect to 3 complex structures $(I, J, K)$ satisfying the quaternion relations, and

$$\Omega_I = \lambda(\omega_J + \sqrt{-1}\omega_K), \lambda \in \mathbb{C}^\times,$$

where $\omega_I, \omega_J, \omega_K$ are the Kähler forms.
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where $\omega_I, \omega_J, \omega_K$ are the Kähler forms.

For $\Omega_I := \frac{1}{2}(-\omega_K + \sqrt{-1}\omega_J), \Omega_J := \frac{1}{2}(-\omega_K - \sqrt{-1}\omega_I)$ we have

1. $\text{Re}(\Omega_I) = \text{Re}(\Omega_J) = \Omega (= -\frac{1}{2}\omega_K)$;

2. $\text{Im}(\Omega_I) = \Omega \circ I, \quad \text{Im}(\Omega_J) = \Omega \circ J$ for $I, J$ integrable almost complex structures;

3. $-2\left(\text{Im}(\Omega_J)\right)^{1,1} = \omega_I > 0$. 
Non-degenerate GK structures: Examples

Suppose $X = (M, I)$ is holomorphic symplectic and CY.

**Fact (Bogomolov)**

Any Calabi–Yau metric $g_{CY}$ on $X$ is hyper-Kähler, i.e. $g_{CY}$ is Kähler with respect to 3 complex structures $(I, J, K)$ satisfying the quaternion relations, and

$$\Omega_I = \lambda(\omega_J + \sqrt{-1}\omega_K), \lambda \in \mathbb{C}^\times,$$

where $\omega_I, \omega_J, \omega_K$ are the Kähler forms.

**Example**

If $(M^{4n}, g_{CY}, I, J, K)$ is a hyper-Kähler manifold, then

$$\Omega_I := \frac{1}{2}(-\omega_K + \sqrt{-1}\omega_J), \Omega_J := \frac{1}{2}(-\omega_K - \sqrt{-1}\omega_I)$$

defines a non-degenerate GK structure on $M$ with $g = g_{CY}$. 
Non-degenerate GK structures: Examples

Lemma (Joyce’s deformation)

$(M, \Omega_I, \Omega_J)$ compact non-degenerate GK mfd,
$\Omega = \text{Re}(\Omega_I) = \text{Re}(\Omega_J)$ a real symplectic form.

- $f \in C^\infty(M)$ gives rise to a Hamiltonian vector field
  $X_f = \Omega^{-1}(df)$ whose flow $\phi^f_t$ satisfy $(\phi^f_t)^*\Omega = \Omega$. 

Example

If $(M, g_{\text{CY}}, I, J, K)$ is a hyper-Kähler manifold, then it admits
many non-Kähler GK metrics.
Non-degenerate GK structures: Examples

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  \((\phi_t^f)^* \Omega = \Omega\).

\(\Rightarrow\) \((\Omega_I, \Omega_{J_t} := (\phi_t^f)^*(\Omega_J))\) satisfy the conditions (1), (2), (3) 
  for \(|t| < \varepsilon\).
Non-degenerate GK structures: Examples

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Example

If \((M^{4n}, g_{\text{CY}}, I, J, K)\) is a hyper-Kähler manifold, then it admits many non-Kähler GK metrics.
Non-degenerate GK structures: Conceptual picture

$(M^{4n}, \Omega_I, \Omega_J)$ a compact non-degenerate GK mfd,

$\Omega = \text{Re}(\Omega_I) = \text{Re}(\Omega_J)$, $G = \text{Ham}(M, \Omega) = \left\langle \phi^f, f \in C^\infty(M) \right\rangle$ the group of Hamiltonian diffeomorphisms
Non-degenerate GK structures: Conceptual picture

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- \(G \times G\) acts on \((\Omega_I, \Omega_J)\) preserving (1) and (2) and \textbf{locally} (3):

\[
K_{(\Omega_I, \Omega_J)}^c := \left\{ (\phi^*(\Omega_I), \psi^*(\Omega_J)), (\phi, \psi) \in G \times G : 
\begin{align*}
(\phi^*(\Omega_I), \psi^*(\Omega_J)) &\text{ satisfy } (3) 
\end{align*}
\right\}.
\]
Non-degenerate GK structures: Conceptual picture

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- \(G \times G\) acts on \((\Omega_I, \Omega_J)\) preserving (1) and (2) and \textbf{locally} (3):

\[
K^c_{(\Omega_I, \Omega_J)} := \left\{ (\phi^*(\Omega_I), \psi^*(\Omega_J)), (\phi, \psi) \in G \times G : \right. \\
\left. (\phi^*(\Omega_I), \psi^*(\Omega_J)) \text{ satisfy } (3) \right\}.
\]

- \(G_d = \{(\phi, \phi) \in G \times G : \phi \in G\}\) acts \textbf{globally} on \(K^c_{(\Omega_I, \Omega_J)}\) and

\[
K_{(\Omega_I, \Omega_J)} := K^c_{(\Omega_I, \Omega_J)}/G_d \approx \left\{ (\Omega_I, \phi^*(\Omega_J)) : (\Omega_I, \phi^*(\Omega_J)) \text{ satisfies } (3) \right\}.
\]
Non-degenerate GK structures: Conceptual picture

Theorem 1 (Streets–A.)

$K^c_{(\Omega_I, \Omega_J)}$ has a formal symplectic structure $\Omega$ such that $G_d$ acts with moment map

$$\mu(\Omega_{I'}, \Omega_{J'}) = \left(\text{Im}(\Omega_{I'} - \Omega_{J'})\right)^{2n} - \lambda \left(\text{Im}(\Omega_{I'} + \Omega_{J'})\right)^{2n},$$

where $\lambda := \frac{\int_M (\Omega_I - \Omega_J)^{2n}}{\int_M (\Omega_I + \Omega_J)^{2n}}$ is a topological constant.
Non-degenerate GK structures: Conceptual picture

Conjecture (GIT package)

$K_{(\Omega_I, \Omega_J)}^c$ admits a unique up to the action of $G_d$ pair $(\Omega_I', \Omega_J')$ such that

$$
\mu(\Omega_I', \Omega_J') = \left( \text{Im}(\Omega_I' - \Omega_J') \right)^{2n} - \lambda \left( \text{Im}(\Omega_I' + \Omega_J') \right)^{2n} = 0
$$
Non-degenerate GK structures: Conceptual picture

Conjecture (GIT package)

$K^c_{(\Omega_I, \Omega_J)}$ admits a unique up to the action of $G_d$ pair $(\Omega_{I'}, \Omega_{J'})$ such that

\[ \mu(\Omega_{I'}, \Omega_{J'}) = \left( \text{Im}(\Omega_{I'} - \Omega_{J'}) \right)^{2n} - \lambda \left( \text{Im}(\Omega_{I'} + \Omega_{J'}) \right)^{2n} = 0 \]

Equivalently, there exists a unique non-degenerate GK structure $(\Omega_I, \phi^*(\Omega_J)) \in K_{(\Omega_I, \Omega_J)}$ ($\phi \in \text{Ham}(M, \Omega)$) such that

\[ \Phi := \frac{\left( \text{Im}(\Omega_I - \Omega_{J'}) \right)^{2n}}{\left( \text{Im}(\Omega_I + \Omega_{J'}) \right)^{2n}} = \lambda. \]
Non-degenerate GK structures: Geometric picture

Lemma (Streets–A.)

Let $(\Omega_I, \Omega_J)$ correspond to the GK structure $(g, I, J, b)$ and

$$\Phi = \frac{\left(\text{Im}(\Omega_I - \Omega_J)\right)^{2n}}{\left(\text{Im}(\Omega_I + \Omega_J)\right)^{2n}}.$$  

Then $\rho^{B,I} = -\sqrt{-1} \partial_J \bar{\partial}_J \Phi$ and $\rho^{B,J} = -\sqrt{-1} \partial_I \bar{\partial}_I \Phi$ are the Ricci forms of the Bismut connections $\nabla^{B,I}$ and $\nabla^{B,J}$ of $(g, I)$ and $(g, J)$. 
Non-degenerate GK structures: Geometric picture

Lemma (Streets–A.)

Let \((\Omega_I, \Omega_J)\) correspond to the GK structure \((g, I, J, b)\) and

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\Phi = \frac{\left(\text{Im}(\Omega_I - \Omega_J)\right)^{2n}}{\left(\text{Im}(\Omega_I + \Omega_J)\right)^{2n}}.
\]

Then \(\rho^{B,I} = -\sqrt{-1} \partial J \bar{\partial} J \Phi\) and \(\rho^{B,J} = -\sqrt{-1} \partial I \bar{\partial} I \Phi\) are the Ricci forms of the Bismut connections \(\nabla^{B,I}\) and \(\nabla^{B,J}\) of \((g, I)\) and \((g, J)\). If \(M\) is compact, \(\Phi = \lambda \iff \rho^{B,I} = \rho^{B,J} = 0\).
Non-degenerate GK structures: Geometric picture

Corollary (Alexandrov–Ivanov, Ivanov–Papadopoulos)

A compact non-degenerate GK mfd \((M, \Omega_I, \Omega_J)\) satisfies \(\Phi = \lambda \iff (\Omega_I, \Omega_J)\) is hyper-Kähler \((g = g_{\text{CY}})\).
Non-degenerate GK structures:
Geometric picture

Corollary (Alexandrov–Ivanov, Ivanov–Papadopoulos)

A compact non-degenerate GK mfd \((M, \Omega_I, \Omega_J)\) satisfies
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Conjecture (Calabi–Yau Conjecture for non-degenerate GK structures)

Let \((M, \Omega_I, \Omega_J)\) be a compact non-degenerate GK manifold. Then
\(\exists! (\Omega_I, \phi^*(\Omega_J))\) \((\phi \in \text{Ham}(\Omega))\) which corresponds to a
hyper-Kähler structure.
Non-degenerate GK structures: Geometric picture

Corollary (Alexandrov–Ivanov, Ivanov–Papadopoulos)

A compact non-degenerate GK mfd $(M,\Omega_I,\Omega_J)$ satisfies
\[ \Phi = \lambda \iff (\Omega_I,\Omega_J) \text{ is hyper-Kähler } (g = g_{CY}). \]

Conjecture (Calabi–Yau Conjecture for non-degenerate GK structures)

Let $(M,\Omega_I,\Omega_J)$ be a compact non-degenerate GK manifold. Then
\[ \exists! (\Omega_I,\phi^*(\Omega_J)) \ (\phi \in \text{Ham}(\Omega)) \text{ which corresponds to a hyper-Kähler structure.} \]
\[ \iff \text{each non-degenerate GK structure is obtained from the Joyce construction and } K_{(\Omega_I,\Omega_J)} \text{ is a (non-abelian) analog of a Kähler class.} \]
Non-degenerate GK structures: Proving the conjecture

Recall:

**Theorem (Cao)**

Let $(X^m, \Theta)$ be a CY manifold. Then, for any Kähler metric $\omega_0$ the solution to the Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega_t = -2\rho_{\omega_t}, \quad (\omega_t)|_{t=0} = \omega_0$$

exists for all $t \in [0, +\infty)$, $\omega_t \in K[\omega_0]$ and $\lim_{t \to \infty} \omega_t = \omega_{CY}$ in $C^\infty$. 
Recall:

**Theorem (Cao)**

Let $(X^m, \Theta)$ be a CY manifold. Then, for any Kähler metric $\omega_0$ the solution to the Kähler-Ricci flow

\[
\frac{\partial}{\partial t} \omega_t = -2\rho_\omega, \quad (\omega_t)|_{t=0} = \omega_0
\]

exists for all $t \in [0, +\infty)$, $\omega_t \in K[\omega_0]$ and $\lim_{t \to \infty} \omega_t = \omega_{CY}$ in $C^\infty$.

Main tool is the reduction to a parabolic Monge-Ampère PDE:

\[
\frac{\partial}{\partial t} \varphi_t = 2 \log \left( \frac{\omega_m^m}{\Theta \wedge \Theta} \right) = MA(\varphi_t), \quad \varphi_t \in K[\omega_0].
\]
Non-degenerate GK structures: Proving the conjecture

**Theorem (Streets–Tian)**

Let $(M, g, I, J, b)$ be a compact GK manifold. Then, the solution $\omega_t = g_t I$ to the generalized Kähler Ricci flow

$$\frac{\partial}{\partial t} \omega_t = -2(\rho_{\omega_t}^B, I)^{1,1}, \quad (\omega_t)|_{t=0} = \omega_I (= gI)$$

exists for $t \in [0, T_{\text{max}})$ and $\exists (J_t, b_t)$ s.t. $(g_t, I, J_t, b_t)$ is GK.
Non-degenerate GK structures: Proving the conjecture

Theorem (Streets–Tian)

Let \((M, g, l, J, b)\) be a compact GK manifold. Then, the solution \(\omega_t = g_t l\) to the generalized Kähler Ricci flow

\[
\frac{\partial}{\partial t} \omega_t = -2(\rho^{B,L}_{\omega_t})_{1,1}^{1,1}, \quad (\omega_t)|_{t=0} = \omega_l (= g_l)
\]

exists for \(t \in [0, T_{\text{max}})\) and \(\exists (J_t, b_t)\) s.t. \((g_t, l, J_t, b_t)\) is GK.

This is a parabolic system (not a single PDE) so there is no \(C^\alpha\) (deGiorgi–Nash–Moser/Krylov–Safonov) estimate nor \(C^{2,\alpha}\) (Evans–Krylov) estimate...
Non-degenerate GK structures:
Proving the conjecture

Theorem 2 (Streets–A.)

Let $(M, \Omega_I, \Omega_J)$ be a compact non-degenerate GK mdf and $(g_t, I, J_t, b_t)$ the solution of the GK Ricci flow starting from $(\Omega_I, \Omega_J)$. Then $(g_t, I, J_t, b_t)$ corresponds to $(\Omega_I, \Omega_{J_t}) \in K_{(\Omega_I, \Omega_J)}$ where $\Omega_{J_t} = \phi^*_t(\Omega_J)$ for $\phi_t$ being the hamiltonian isotopy generated by the momentum $\Phi_t$. 
Non-degenerate GK structures: Proving the conjecture

Theorem 2 (Streets–A.)

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\[
\frac{\partial}{\partial t} \Phi_t = -\Delta_{g_t} \Phi_t
\]
Non-degenerate GK structures:
Proving the conjecture

Using the maximum principle for

$$\frac{\partial}{\partial t} \Phi_t = -\Delta_{g_t} \Phi_t$$

Corollary (New a priori estimates)

Let $(M, \Omega_I, \Omega_J)$ be a compact non-degenerate GK mdf and $(g_t, I_t, J_t, b_t), t \in [0, T_{\text{max}})$ the solution of the GK Ricci flow starting from $(\Omega_I, \Omega_J)$. Then

$$\sup_{M \times [0, T_{\text{max}})} |\Phi_t| \leq \sup_{M \times \{0\}} |\Phi_0|$$

$$\sup_{M \times \{t\}} |\nabla \Phi_t|^2 \leq t^{-1} \left( \sup_{M \times \{0\}} |\Phi_0|^2 \right)$$
Non-degenerate GK structures:
Proving the conjecture

Using the maximum principle for

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\frac{\partial}{\partial t} \Phi_t = -\Delta_{g_t} \Phi_t
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Corollary (New a priori estimates)

Let \((M, \Omega_I, \Omega_J)\) be a compact non-degenerate GK mdf and
\((g_t, I, J_t, b_t), t \in [0, T_{\text{max}})\) the solution of the GK Ricci flow
starting from \((\Omega_I, \Omega_J)\). Then

\[
\sup_{M \times [0, T_{\text{max}}]} |\Phi_t| \leq \sup_{M \times \{0\}} |\Phi_0| \Rightarrow \omega_t^{2n} \leq C \omega_0^{2n}, \ |b_t|^2 \leq C
\]

\[
\sup_{M \times \{t\}} |\nabla \Phi_t|^2 \leq t^{-1} \left( \sup_{M \times \{0\}} |\Phi_0|^2 \right) \Rightarrow \lim_{t \to \infty} \Phi_t = \lambda
\]
Non-degenerate GK structures:
Using the a priori estimates

Theorem 3 (Streets–A.)

Let \((M, g_0, I, J_0, b_0)\) be a compact non-degenerate GK mdf and \((g_t, I, J_t, b_t), t \in [0, T_{\text{max}})\) the solution of the GK Ricci flow. Suppose there exits a uniform constant \(C > 0\) s.t.

\[
\frac{1}{C} g_0 \leq g_t \leq C g_0,
\]

Then \(T_{\text{max}} = \infty\), \(\lim_{t \to \infty} g_t = g_\infty\) in \(C^\infty\) and \((g_\infty, I, J_\infty, b_\infty)\) is Hyper-Kähler with \(J_\infty = \phi^*_\infty (J_0), \phi_\infty \in \text{Ham}(M, \Omega)\).
Non-degenerate GK structures:
Using the a priori estimates

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Let \((M, g_0, I, J_0, b_0)\) be a compact non-degenerate GK mdf and \((g_t, I, J_t, b_t), t \in [0, T_{\text{max}})\) the solution of the GK Ricci flow. Suppose there exists a uniform constant \(C > 0\) s.t.

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\]

Then \(T_{\text{max}} = \infty\), \(\lim_{t \to \infty} g_t = g_\infty\) in \(C^\infty\) and \((g_\infty, I, J_\infty, b_\infty)\) is Hyper-Kähler with \(J_\infty = \phi_\infty^*(J_0), \phi_\infty \in \text{Ham}(M, \Omega)\).  
⇒ the Calabi–Yau conjecture for non-degenerate GK holds true.
Non-degenerate GK structures: Using the a priori estimates

Theorem 4 (Streets–A.)

Let \((M, g_0, I, J_0, b_0)\) be a compact non-degenerate GK mdf and \((g_t, I, J_t, b_t), t \in [0, T_{\text{max}})\) the solution of the GK Ricci flow. Suppose \((M, I)\) is CY. Then there exists a constant \(C = C(T_{\text{max}}) > 0\) s.t.

\[
\frac{1}{C} g_0 \leq g_t \leq C g_0,
\]
Non-degenerate GK structures: Using the a priori estimates

Theorem 4 (Streets–A.)

Let \((M, g_0, I, J_0, b_0)\) be a compact non-degenerate GK mdf and \((g_t, I, J_t, b_t), t \in [0, T_{\text{max}})\) the solution of the GK Ricci flow. Suppose \((M, I)\) is CY. Then there exists a constant \(C = C(T_{\text{max}}) > 0\) s.t.

\[
\frac{1}{C} g_0 \leq g_t \leq C g_0,
\]

\[\Rightarrow T_{\text{max}} = \infty, \lim_{t \to \infty} \omega_t = \omega_\infty \text{ where } \omega_\infty \text{ is a closed } (1, 1) \text{ current on } (M, I).\]
THANK YOU !