The Generalized Kähler Geometry of Holomorphic Symplectic Manifolds

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MDS
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Plan of the talk

1. **The Calabi program and Calabi–Yau manifolds**
   - E. Calabi, *On Kähler manifolds with vanishing canonical class*, Princeton University Press, Mathematical Series, 1957;
   - Y.-S. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. National Acad. Sci. U.S. A., 1977.
2. **Generalized Kähler geometry**
   - S. Gates, J. Hull, M. Rocek, *Twisted multiplets and new supersymmetric nonlinear σ-models*. Nuclear Phys. B, 1984;
   - N. Hitchin, *Generalized Calabi-Yau manifolds*. Q. J. Math., 2003;
   - M. Gualtieri, *Generalized Kähler geometry*, Comm. Math. Phys., 2014.
3. **Calabi–Yau conjecture in generalized Kähler geometry**
   joint work with Jeff Streets (UCI): arXiv:1703.08650.

Calabi–Yau complex manifolds

Definition (Calabi–Yau manifold)

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with $\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)(z) > 0$,

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$$\Theta = \theta(z) dz_1 \wedge \cdots \wedge dz_m$$

  with $\theta(z)$ holomorphic and $\theta(z) \neq 0$.

Calabi–Yau complex manifolds

Examples of CY manifolds

- (tori) $X^m = \mathbb{C}^m/(\mathbb{Z}^m \oplus \sqrt{-1}\mathbb{Z}^m) = T^m_{\mathbb{C}}$ with

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- $X^m \subset \mathbb{P}^{m+1}$ of degree $m+2$ is CY (X is projective with $K_X = \mathcal{O}$).
- Deforming the complex structure in the above examples leads to CY manifolds: each elliptic complex curve and each $K3$ complex surface is CY.

The Kähler geometry of CY manifolds

Definition (Kähler class)

A **Kähler class** of Kähler metrics on X is the space of smooth functions

$$K_{[\omega_0]} := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\},$$

where ω_0 is a given (reference) Kähler metric.

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Example ($\mathrm{dim}_{\mathbb{C}} X = 1$)

Any riemannian metric is determined in a holomorphic chart by $\omega_0 = \sqrt{-1}h(z)dz \wedge d\bar{z}, h > 0$.

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$$\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \Longleftrightarrow \Delta_{\omega_0}\varphi = (1 - e^\psi) \Longleftrightarrow \int_X \omega = \int_X \omega_0.$$

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$$K_{[\omega_0]} = \{\text{volume normalized conformal class on } X\}.$$

The Kähler geometry of CY manifolds

Definition (Ricci form)

The **Ricci form** of a Kähler metric ω on X is

$$\rho_\omega := \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\Theta\wedge\bar{\Theta}}{\omega^m}\right)$$

where Θ is any local holomorphic $(m, 0)$-form.

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Any riemannian metric is determined in a holomorphic chart by $\omega = \sqrt{-1}h(z)dz\wedge d\bar{z}$, $h>0$. For $\Theta = dz$ we get

$$\rho_\omega = K(z)\omega$$

where $K(z) = -\frac{1}{h(z)}\Delta_0 \log h(z)$ is the Gauss curvature.

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Theorem (Yau)

Let (X^m,Θ) be a CY manifold. Then, $\exists$ unique

$$\omega_{\mathrm{CY}} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \in K_{[\omega_0]}$$

such that

$$\rho_{\omega_{\mathrm{CY}}} = 0$$

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such that

$$\rho_{\omega_{\mathrm{CY}}} = 0 \iff (\omega_{\mathrm{CY}})^{\wedge m} = \lambda(\Theta\wedge\bar{\Theta}), \lambda = const.$$

The Kähler geometry of CY manifolds

Definition (Kähler Ricci flow)

The **Kähler Ricci flow** starting from a Kähler metric ω_0 on X is any smooth family of Kähler metrics ω_t solving the geometric PDE

$$\frac{\partial}{\partial t}\omega_t = -2\rho_{\omega_t}, \ (\omega_t)_{|t=0} = \omega_0.$$

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Theorem (Cao)

Let (X^m, Θ) be a CY manifold. Then, for any Kähler metric ω_0 the solution to the Kähler-Ricci flow exists for all $t \in [0, +\infty)$, $\omega_t \in K_{[\omega_0]}$ and $\lim_{t\to\infty}\omega_t = \omega_{\mathrm{CY}}$ in C^∞.

The Kähler geometry of CY manifolds
Summary: The Calabi Program

- The Kähler geometry is described in terms of Kähler classes K_α where $\alpha = [\omega_0] \in H^2_{\mathrm{dR}}(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{C})$ runs over the **Kähler cone**.
- (uniqueness) Each Kähler class K_α contains a unique canonical representative $\omega_{\mathrm{CY},\alpha}$ and any other Kähler metric $\omega \in K_\alpha$ is written

$$\omega = \omega_{\mathrm{CY},\alpha} + \sqrt{-1}\partial\bar{\partial}\varphi, \ \varphi \in C^\infty(X).$$

- (connectedness) The Kähler Ricci flow allows one to reach the canonical representative $\omega_{\mathrm{CY},\alpha}$.

Generalized Kähler geometry

(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Definition (GK structure)

A **generalized Kähler structure** (GK) on a (real) $2m$-dimensional manifold $M^{2m}_{\mathbb{R}}$ is defined by the data (I, J, g, b) where:

- I and J are two complex structures on $M^{2m}_{\mathbb{R}}$;

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- b is a 2-form;
- a first order compatibility relation

$$\partial_I \omega_I = \sqrt{-1}\bar{\partial}_I(b_I^{2,0}), \quad \partial_J \omega_J = -\sqrt{-1}\bar{\partial}_J(b_J^{2,0}),$$

where $\omega_I = gI, \omega_J = gJ$ are the Kähler forms.

Generalized Kähler geometry

(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Example (trivial)

$X = (M^{2m}, I)$ a complex manifold and (g, ω_I) a Kähler metric. Then, letting $J := -I$, $b := 0$ we obtain a GK structure (g, I, J, b).

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Problem

Are there non-trivial examples?

Generalized Kähler geometry

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Theorem (Gauduchon–Grantcharov–A. for $m = 2$; Hitchin for $m \geq 2$)

Let (M^{2m}, g, I, J, b) *be GK and* $\sigma := (IJ - JI)g^{-1} \in \Gamma(\wedge^2 TM)$. *Then* $\sigma_I := \sigma - \sqrt{-1}(I\sigma) \in H^0(M, \wedge^2(T_I^{1,0}M))$ *is* **Holomorphic Poisson** *on* $X = (M, I)$,

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$$\sigma_I = \frac{1}{2}\sum_{i,j=1}^{m} \sigma_{ij}(z)\Big(\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}\Big), \ (\sigma_{ij} = -\sigma_{ji})$$

with $\sigma_{ij}(z)$ *holomorphic, and*

$$[\sigma_I, \sigma_I] = 0 \Leftrightarrow \sum_{\ell=1}^{m}\Big(\sum_{(ijk)\in S_3} \sigma_{i\ell}(z)\frac{\partial \sigma_{jk}}{\partial z_\ell}(z)\Big) = 0.$$

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Corollary (Gualtieri-A.)

If X is complex surface of general type not covered by $\mathbb{D} \times \mathbb{D}$, then $\nexists$ non-trivial GK structures.

Generalized Kähler geometry

(after Gates–Hull–Rocek, Hitchin, Gualtieri)

Some basic open problems

Let $X = (M^{2m}, I)$ be a compact complex manifold and $\sigma_I \neq 0$ is a holomorphic Poisson structure.

- Is there a non-trivial GK structure (g, I, J, b) with $\sigma = (IJ - JI)g^{-1} = \mathrm{Re}(\sigma_I)$?
  True if $m = 2$ (Goto) or if (X, σ_I) is a toric variety (Boulanger).
- If (X, σ_I) admits a compatible GK structure does $X = (M, I)$ admit a Kähler metric?
  True if $m = 2$ (Gauduchon–Grantcharov–A., Gualtieri–A.)
- Describe the GK geometry of (X, σ_I) in a similar way as we described the Kähler geometry of a CY manifold.

Non-degenerate GK structures

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The GK structure (g, I, J, b) on M^{2m} is called **non-degenerate** if the holomorphic Poisson structure

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- $\Longleftrightarrow \sigma_I : T^*_X \cong T_X$ where $X = (M, I)$;
- $\Longleftrightarrow \sigma : T^*_M \cong T_M$;
- $\Longleftrightarrow \sigma_J : T^*_Y \cong T_Y$ where $Y = (M, J)$.

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is non-degenerate, i.e. $\mathrm{det}_{\mathbb{C}}(\sigma_{ij}(z)) \neq 0$

- $\iff \Omega_I = \sigma_I^{-1}$ is closed and non-degenerate $(2,0)$-form on X,
- $\iff \Omega = \sigma^{-1} = \mathrm{Re}(\Omega_I)$ a closed and non-degenerate real 2-form on M,
- $\iff \Omega_J = \sigma_J^{-1}$ is closed and non-degenerate $(2,0)$-form on Y.

Non-degenerate GK structures: revisited

Lemma (Reduction of non-degenerate GK structures)

On M^{4n} we have a bijection

$$\{\mathrm{non-degenerate\ GK\ structures}\} \longleftrightarrow \{(\Omega_I, \Omega_J)\}$$

where Ω_I, Ω_J are closed complex-valued 2-forms satisfying

(1) $\mathrm{Re}(\Omega_I) = \mathrm{Re}(\Omega_J) = \Omega$ *is a real symplectic form;*

(2) $\mathrm{Im}(\Omega_I) = \Omega \circ I,\ \mathrm{Im}(\Omega_J) = \Omega \circ J$ *for I, J integrable almost complex structures;*

(3) $\omega_I := -2\Big(\mathrm{Im}(\Omega_J)\Big)^{1,1}_I > 0$ $(g = -\omega_I \circ I)$.

Holomorphic symplectic manifolds

Definition (holomorphic symplectic manifold)

A **holomorphic symplectic** manifold is a smooth, compact, complex $m = 2n$ dimensional manifold $X^{2n} = (M^{4n}, I)$ which admits a closed non-degenerate $(2, 0)$-form Ω_I.

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In a holomorphic chart:

$$\Omega_I = \frac{1}{2} \sum_{i,j=1}^{2n} \omega_{ij}(z) dz_i \wedge dz_j, \quad (\omega_{ij} = -\omega_{ji})$$

with $\omega_{ij}(z)$ holomorphic functions, s.t. $\det_{\mathbb{C}}(\omega_{ij}(z)) \neq 0$ and

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Fact

If (X, Ω_I) is holomorphic symplectic then

$$\Theta := (\Omega_I)^{\wedge n} = \left(\det_{\mathbb{C}}(\omega_{ij}(z))\right)^{\frac{1}{2}} dz_1 \wedge \cdots \wedge dz_{2n}$$

trivializes K_X, i.e. X is CY if it admits a Kähler metric.

Non-degenerate GK structures: Examples

Suppose $X = (M, I)$ is holomorphic symplectic and CY.

Fact (Bogomolov)

Any Calabi–Yau metric g_{CY} on X is hyper-Kähler, i.e. g_{CY} is Kähler with respect to 3 complex structures (I, J, K) satisfying the quaternion relations, and

$$\Omega_I = \lambda(\omega_J + \sqrt{-1}\omega_K), \lambda \in \mathbb{C}^\times,$$

where $\omega_I, \omega_J, \omega_K$ are the Kähler forms.

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For $\Omega_I := \frac{1}{2}\big(-\omega_K + \sqrt{-1}\omega_J\big), \Omega_J := \frac{1}{2}\big(-\omega_K - \sqrt{-1}\omega_I\big)$ we have

(1) $\mathrm{Re}(\Omega_I) = \mathrm{Re}(\Omega_J) = \Omega\ (= -\frac{1}{2}\omega_K)$;

(2) $\mathrm{Im}(\Omega_I) = \Omega \circ I,\ \mathrm{Im}(\Omega_J) = \Omega \circ J$ for I, J integrable almost complex structures;

(3) $-2\Big(\mathrm{Im}(\Omega_J)\Big)^{1,1}_I = \omega_I > 0$.

Non-degenerate GK structures: Examples

Suppose $X = (M, I)$ is holomorphic symplectic and CY.

Fact (Bogomolov)

Any Calabi–Yau metric g_{CY} on X is hyper-Kähler, i.e. g_{CY} is Kähler with respect to 3 complex structures (I, J, K) satisfying the quaternion relations, and

$$\Omega_I = \lambda(\omega_J + \sqrt{-1}\omega_K), \lambda \in \mathbb{C}^\times,$$

where $\omega_I, \omega_J, \omega_K$ are the Kähler forms.

Example

If $(M^{4n}, g_{\mathrm{CY}}, I, J, K)$ is a hyper-Kähler manifold, then

$$\Omega_I := \frac{1}{2}\big(- \omega_K + \sqrt{-1}\omega_J\big), \Omega_J := \frac{1}{2}\big(- \omega_K - \sqrt{-1}\omega_I\big)$$

defines a non-degenerate GK structure on M with $g = g_{\mathrm{CY}}$.

Non-degenerate GK structures: Examples

Lemma (Joyce's deformation)

(M, Ω_I, Ω_J) *compact non-degenerate GK mfd,*
$\Omega = \mathrm{Re}(\Omega_I) = \mathrm{Re}(\Omega_J)$ *a real symplectic form.*

- $f \in C^\infty(M)$ *gives rise to a* **Hamiltonian vector field** $X_f = \Omega^{-1}(df)$ *whose flow* ϕ_t^f *satisfy* $(\phi_t^f)^*\Omega = \Omega$*.*

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- $\implies (\Omega_I, \Omega_{J_t} := (\phi_t^f)^*(\Omega_J))$ *satisfy the conditions* $(1), (2), (3)$ *for* $|t| < \varepsilon$.

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- *$\Longrightarrow (\Omega_I, \Omega_{J_t} := (\phi_t^f)^*(\Omega_J))$ satisfy the conditions (1), (2), (3) for $|t| < \varepsilon$.*

Example

If $(M^{4n}, g_{\mathrm{CY}}, I, J, K)$ is a hyper-Kähler manifold, then it admits many non-Kähler GK metrics.

Non-degenerate GK structures: Conceptual picture

$(M^{4n}, \Omega_I, \Omega_J)$ a compact non-degenerate GK mfd,
$\Omega = \mathrm{Re}(\Omega_I) = \mathrm{Re}(\Omega_J)$, $G = \mathrm{Ham}(M, \Omega) = \left\langle \phi_1^f, f \in C^\infty(M) \right\rangle$ the group of Hamiltonian diffeomorphisms

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- $G \times G$ acts on (Ω_I, Ω_J) preserving (1) and (2) and **locally** (3):

$$K^c_{(\Omega_I,\Omega_J)} := \Big\{ \big(\phi^*(\Omega_I), \psi^*(\Omega_J)\big), (\phi, \psi) \in G \times G : \big(\phi^*(\Omega_I), \psi^*(\Omega_J)\big) \ \text{satisfy } (3) \Big\}.$$

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- $G_d = \{(\phi, \phi) \in G \times G : \phi \in G\}$ acts **globally** on $K^c_{(\Omega_I,\Omega_J)}$ and

$$K_{(\Omega_I,\Omega_J)} := K^c_{(\Omega_I,\Omega_J)}/G_d \\ \cong \Big\{ \big(\Omega_I, \phi^*(\Omega_J)\big) : \big(\Omega_I, \phi^*(\Omega_J)\big) \text{ satisfies (3)} \Big\}.$$

Non-degenerate GK structures: Conceptual picture

Theorem 1 (Streets–A.)

$K^c_{(\Omega_I,\Omega_J)}$ has a formal symplectic structure $\mathbf{\Omega}$ such that G_d acts with moment map

$$\boldsymbol{\mu}(\Omega_{I'},\Omega_{J'}) = \Big(\mathrm{Im}(\Omega_{I'}-\Omega_{J'})\Big)^{2n} - \lambda\Big(\mathrm{Im}(\Omega_{I'}+\Omega_{J'})\Big)^{2n},$$

where $\lambda := \frac{\int_M(\Omega_I-\Omega_J)^{2n}}{\int_M(\Omega_I+\Omega_J)^{2n}}$ is a topological constant.

Non-degenerate GK structures: Conceptual picture

Conjecture (GIT package)

$K^c_{(\Omega_I,\Omega_J)}$ admits a unique up to the action of G_d pair $(\Omega_{I'},\Omega_{J'})$ such that

$$\boldsymbol{\mu}(\Omega_{I'},\Omega_{J'}) = \Big(\mathrm{Im}(\Omega_{I'} - \Omega_{J'})\Big)^{2n} - \lambda\Big(\mathrm{Im}(\Omega_{I'} + \Omega_{J'})\Big)^{2n} = 0$$

Non-degenerate GK structures: Conceptual picture

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Equivalently, there exists a unique non-degenerate GK structure $(\Omega_I,\phi^*(\Omega_J)) \in K_{(\Omega_I,\Omega_J)}$ ($\phi \in \mathrm{Ham}(M,\Omega)$) such that

$$\Phi := \frac{\Big(\mathrm{Im}(\Omega_I-\Omega_{J'})\Big)^{2n}}{\Big(\mathrm{Im}(\Omega_I+\Omega_{J'})\Big)^{2n}} = \lambda.$$

Non-degenerate GK structures: Geometric picture

Lemma (Streets–A.)

Let (Ω_I, Ω_J) correspond to the GK structure (g, I, J, b) and

$$\Phi = \frac{\Big(\mathrm{Im}(\Omega_I - \Omega_J)\Big)^{2n}}{\Big(\mathrm{Im}(\Omega_I + \Omega_J)\Big)^{2n}}.$$

Then $\rho^{B,I} = -\sqrt{-1}\partial_J\bar{\partial}_J\Phi$ and $\rho^{B,J} = -\sqrt{-1}\partial_I\bar{\partial}_I\Phi$ are the Ricci forms of the Bismut connections $\nabla^{B,I}$ and $\nabla^{B,J}$ of (g, I) and (g, J).

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Non-degenerate GK structures: Geometric picture

Corollary (Alexandrov–Ivanov, Ivanov–Papadopoulos)

A compact non-degenerate GK mfd (M, Ω_I, Ω_J) satisfies $\Phi = \lambda \Leftrightarrow (\Omega_I, \Omega_J)$ is hyper-Kähler ($g = g_{\mathrm{CY}}$).

Non-degenerate GK structures: Geometric picture

Corollary (Alexandrov–Ivanov, Ivanov–Papadopoulos)

A compact non-degenerate GK mfd (M, Ω_I, Ω_J) *satisfies* $\Phi = \lambda \Leftrightarrow (\Omega_I, \Omega_J)$ *is hyper-Kähler* ($g = g_{\mathrm{CY}}$).

Conjecture (Calabi–Yau Conjecture for non-degenerate GK structures)

Let (M, Ω_I, Ω_J) be a compact non-degenerate GK manifold. Then $\exists!$ $(\Omega_I, \phi^*(\Omega_J))$ $(\phi \in \mathrm{Ham}(\Omega))$ which corresponds to a hyper-Kähler structure.

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Conjecture (Calabi–Yau Conjecture for non-degenerate GK structures)

Let (M, Ω_I, Ω_J) be a compact non-degenerate GK manifold. Then $\exists!$ $(\Omega_I, \phi^*(\Omega_J))$ $(\phi \in \mathrm{Ham}(\Omega))$ which corresponds to a hyper-Kähler structure.

$\Leftrightarrow$ each non-degenerate GK structure is obtained from the Joyce construction and $K_{(\Omega_I, \Omega_J)}$ is a (**non-abelian**) analog of a Kähler class.

Non-degenerate GK structures: Proving the conjecture

Recall:

Theorem (Cao)

Let (X^m, Θ) *be a CY manifold. Then, for any Kähler metric* ω_0 *the solution to the Kähler-Ricci flow*

$$\frac{\partial}{\partial t}\omega_t = -2\rho_{\omega_t}, \quad (\omega_t)_{|t=0} = \omega_0$$

exists for all $t \in [0, +\infty)$, $\omega_t \in K_{[\omega_0]}$ *and* $\lim_{t\to\infty}\omega_t = \omega_{\mathrm{CY}}$ *in* C^∞.

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Main tool is the reduction to a parabolic Monge-Ampère PDE:

$$\frac{\partial}{\partial t}\varphi_t = 2\log\left(\frac{\omega_{\varphi_t}^m}{\Theta \wedge \bar{\Theta}}\right) = \mathrm{MA}(\varphi_t), \quad \varphi_t \in K_{[\omega_0]}.$$

Non-degenerate GK structures: Proving the conjecture

Theorem (Streets–Tian)

Let (M, g, I, J, b) be a compact GK manifold. Then, the solution $\omega_t = g_t I$ to the **generalized Kähler Ricci flow**

$$\frac{\partial}{\partial t}\omega_t = -2(\rho^{B,I}_{\omega_t})^{1,1}_I, \quad (\omega_t)_{|t=0} = \omega_I (= gI)$$

exists for $t \in [0, T_{\max})$ and $\exists\ (J_t, b_t)$ s.t. (g_t, I, J_t, b_t) is GK.

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This is a parabolic system (not a single PDE) so there is no C^α (deGiorgi–Nash–Moser/Krylov–Safonov) estimate nor $C^{2,\alpha}$ (Evans–Krylov) estimate...

Non-degenerate GK structures: Proving the conjecture

Theorem 2 (Streets–A.)

Let (M, Ω_I, Ω_J) be a compact non-degenerate GK mdf and (g_t, I, J_t, b_t) the solution of the GK Ricci flow starting from (Ω_I, Ω_J). Then (g_t, I, J_t, b_t) corresponds to $(\Omega_I, \Omega_{J_t}) \in K_{(\Omega_I, \Omega_J)}$ where $\Omega_{J_t} = \phi_t^*(\Omega_J)$ for ϕ_t being the hamiltonian isotopy generated by the momentum Φ_t.

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$$\frac{\partial}{\partial t}\Phi_t = -\Delta_{g_t}\Phi_t$$

Non-degenerate GK structures: Proving the conjecture

Using the maximum principle for

$$\frac{\partial}{\partial t}\Phi_t = -\Delta_{g_t}\Phi_t$$

Corollary (New a priori estimates)

Let (M, Ω_I, Ω_J) be a compact non-degenerate GK mdf and $(g_t, I, J_t, b_t), t \in [0, T_{\max})$ the solution of the GK Ricci flow starting from (Ω_I, Ω_J). Then

$$\sup_{M\times[0,T_{\max})} |\Phi_t| \leq \sup_{M\times\{0\}} |\Phi_0|$$

$$\sup_{M\times\{t\}} |\nabla\Phi_t|^2 \leq t^{-1}\Big(\sup_{M\times\{0\}} |\Phi_0|^2\Big)$$

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$$\sup_{M\times[0,T_{\max})} |\Phi_t| \le \sup_{M\times\{0\}} |\Phi_0| \Rightarrow \omega_t^{2n} \le C\omega_0^{2n}, \; |b_t|^2 \le C$$

$$\sup_{M\times\{t\}} |\nabla\Phi_t|^2 \le t^{-1}\Big(\sup_{M\times\{0\}} |\Phi_0|^2\Big) \Rightarrow \lim_{t\to\infty} \Phi_t = \lambda$$

Non-degenerate GK structures: Using the a priori estimates

Theorem 3 (Streets–A.)

Let (M, g_0, I, J_0, b_0) *be a compact* **non-degenerate** *GK mdf and* $(g_t, I, J_t, b_t), t \in [0, T_{\max})$ *the solution of the GK Ricci flow.*
Suppose there exits a uniform constant $C > 0$ *s.t.*

$$\frac{1}{C} g_0 \leq g_t \leq C g_0,$$

Then $T_{\max} = \infty$, $\lim_{t\to\infty} g_t = g_\infty$ *in* C^∞ *and* $(g_\infty, I, J_\infty, b_\infty)$ *is Hyper-Kähler with* $J_\infty = \phi_\infty^*(J_0), \phi_\infty \in \mathrm{Ham}(M, \Omega)$.

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$\Rightarrow$ *the Calabi–Yau conjecture for non-degenerate GK holds true.*

Non-degenerate GK structures: Using the a priori estimates

Theorem 4 (Streets–A.)

Let (M, g_0, I, J_0, b_0) *be a compact* **non-degenerate** *GK mdf and* $(g_t, I, J_t, b_t), t \in [0, T_{\max})$ *the solution of the GK Ricci flow.* *Suppose* (M, I) *is CY. Then there exits a constant* $C = C(T_{\max}) > 0$ *s.t.*

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Non-degenerate GK structures: Using the a priori estimates

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Let (M, g_0, I, J_0, b_0) *be a compact* **non-degenerate** *GK mdf and* $(g_t, I, J_t, b_t), t \in [0, T_{\max})$ *the solution of the GK Ricci flow. Suppose* (M, I) *is CY. Then there exits a constant* $C = C(T_{\max}) > 0$ *s.t.*

$$\frac{1}{C} g_0 \leq g_t \leq C g_0,$$

$\Rightarrow T_{\max} = \infty$, $\lim_{t\to\infty} \omega_t = \omega_\infty$ *where* ω_∞ *is a closed* $(1,1)$ *current on* (M, I).

THANK YOU !