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Representation of Certain Euler-type Sums via Values of the Riemann Zeta Function

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Let $\zeta(z)$ be the Riemann zeta function. This paper, which is a continuation of our recent work [2], is a study of certain Euler-type sums and integrals and their relation to zeta values. It is well-known that $\zeta(2n) = \frac{(-1)^{n+1}B_{2n}}{2(2n)!}(2\pi)^{2n}$ (where B_{2n} are the Bernoulli numbers), and that no similar expression exists at present for the values of the zeta function at odd integers. In a classical work [1], Euler derives the formula

$$\zeta(3) = \frac{2\pi^2}{7}\log 2 + \frac{16}{7}\int_0^{\frac{\pi}{2}} x\log(\sin x)\mathrm{d}x \tag{1}$$

via intricate manipulations of divergent and convergent series. His proof is rather complicated and takes up 16 pages. In [2], we gave a very simple proof (believed to be the simplest so far) of this formula, of the similar equation

$$\zeta(3) = \frac{2\pi^2}{9}\log 2 + \frac{16}{3\pi}\int_0^{\frac{\pi}{2}} x^2\log(\sin x)\mathrm{d}x, \tag{2}$$

and of a theorem which generalizes (1) and (2). After a brief discussion of the necessary preliminaries from [2], we shall present several new results regarding the evaluation of some interesting Euler-type sums in terms of values of the Riemann zeta function.

References

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