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Supperabundant Components of the Hilbert Scheme of Curves Using Double Covers of Irrational Curves

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Let $\mathcal{I}_{d,g,r}$ be the union of irreducible components of the Hilbert scheme whose general points correspond to smooth irreducible non-degenerate complex curves of degree d and genus g embedded in $\mathbb{P}^r$. It can be decomposed into a union of irreducible components as

$$\mathcal{I}_{d,g,r} = R_1 \cup \cdots \cup R_k \cup S_1 \cup \cdots \cup S_l$$

where

(1) the components R_i, $i = 1, \ldots, k$, called *regular*, are characterized by being generically smooth and having the expected dimension $\lambda_{d,g,r} := (r+1)d - (r-3)(g-1)$;

(2) the components S_j, $j = 1, \ldots, l$, called *superabundant*, are characterized by being non-reduced or having dimension greater than $\lambda_{d,g,r}$.

When the *Brill-Noether number* $\rho(d, g, r) := g - (r+1)(g-d+r)$ is positive, $\mathcal{I}_{d,g,r}$ has the unique component dominating the moduli space $\mathcal{M}_g$ of smooth curves of genus g. It is usually referred to as *distinguished* component. It is indeed regular. Severi claimed in [Sev1921] that $\mathcal{I}_{d,g,r}$ is irreducible if $d \geq g + r$. Ein proved the conjecture for $r = 3$ and 4, see [Ein86] and [Ein87], while Mezzetti and Sacchiero [MS1989], and Keem [Kee94] gave examples showing that it doesn't hold for $r \geq 6$. All of these constructions used curves that were $m-$sheeted coverings of $\mathbb{P}^1$ with $m \geq 3$. Ein also showed in [Ein87] that $\mathcal{I}_{d,g,r}$ is irreducible for $d > \frac{2r-2}{r+2}g + \frac{r+8}{r+2}$ and $r \geq 5$. Subsequently, Kim extended in [Kim2001] the irreducibility range to $d > \eta_3 := \frac{2r-4}{r+1}g + \frac{r+13}{r+1}$ for $r \geq 8$. Her work also hinted that for fixed g and r, the scheme $\mathcal{I}_{d,g,r}$ can acquire additional components if $d \leq \eta_3$. The additional components that she found were dominated by trigonal curves if $\frac{2r-6}{r+1}g + \frac{2r+26}{r+1} =: \eta_4 < d \leq \eta_3$ and by 4−gonal curves if $\frac{2r-8}{r+1}g + \frac{3r+43}{r+1} =: \eta_5 < d \leq \eta_4$. In our work we establish the existence of components of $\mathcal{I}_{d,g,r}$ dominated by families of curves that are multi-coverings of irrational curves. To our knowledge, these are the first examples of such kind. For mostly technical reasons, our study is limited to the case $\eta_5 < d \leq \eta_3$. Our main result is that if $\eta_5 < d \leq \min\left\{\left(2-\frac{8}{r}\right)g + \left(2+\frac{8}{r}\right),\ 2g-28\right\}$ the Hilbert scheme $\mathcal{I}_{d,g,r}$ possesses a component dominated by curves that are *double-covers of irrational curves.* It is indeed superabundant, but in some cases it has the minimal possible dimension $\lambda_{d,g,r}$ in addition to being generically smooth, which makes it regular but different from the distinguished one. In our approach we use curves on ruled surfaces and some properties of the Hilbert scheme of scrolls whose base curve has general moduli. The technique allows also to show the existence of components of $\mathcal{I}_{d,g,r}$ that are dominated by triple coverings of irrational curves if $\eta_4 < d \leq \eta_3$ and $3 \mid (2g-2-d)$.

References

[Ein86] L. Ein, *Hilbert scheme of smooth space curves.* Ann. Sci. E'cole Norm. Sup., 19(4) (1986), 469–478.

[Ein87] L. Ein, *The irreducibility of the Hilbert scheme of complex space curves,* In: Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., 46, Part 1, (1987), 83–87, Amer. Math. Soc., Providence, RI.

[Kee94] C. Keem, *Reducible Hilbert scheme of smooth curves with positive Brill-Noether number,* Proc. Amer. Math. Soc., 122(2) (1994), 349–354.

[MS1989] E. Mezzetti, G. Sacchiero, (1989), *Gonality and Hilbert schemes of smooth curves,* In: Algebraic curves and projective geometry (Trento, 1988), Lecture Notes in Math., 1389, Springer, Berlin, 1989, 183–194.

[Kim2001] S. Kim, *On the irreducibility of the family of algebraic curves in complex projective space $\mathbb{P}^r$,* Comm. Algebra, 29(10) (2001), 4321–4331.

[Sev1921] F. Severi, *Vorlesungen über algebraische Geometrie.* Teubner, Leipzig, (1921).