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Bicommutative Algebras

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An algebra over a field K is *right-commutative* if it satisfies the polynomial identity $(x_1x_2)x_3 = (x_1x_3)x_2$. Similarly the identity of *left-commutativity* is $x_1(x_2x_3) = x_2(x_1x_3)$. Algebras which are both right- and left-commutative are *bicommutative*. One-sided commutative algebras appeared already in the paper by Cayley [1] in 1857. In modern language this is the right-symmetric Witt algebra $W_1^{\mathrm{rsym}} = \left\{ f\dfrac{d}{dx} \mid f \in K[x] \right\}$ with multiplication $\left(f_1\dfrac{d}{dx}\right) * \left(f_2\dfrac{d}{dx}\right) = \left(f_2\dfrac{df_1}{dx}\right)\dfrac{d}{dx}$ This algebra is left-commutative and right-symmetric and hence is an example of a Novikov algebra. (Novikov algebras and their opposite were introduced in the 1970s and 1980s by Gel'fand and Dorfman in their study of the Hamiltonian operator in finite-dimensional mechanics and by Balinskii and Novikov in relation with the equations of hydrodynamics.) Examples of bicommutative algebras are the two-dimensional algebras $A_{\pi,\varrho}$, $\pi, \varrho \in K$, generated by an element r and with multiplication rules $r \cdot r^2 = \pi r^2$, $r^2 \cdot r = \varrho r^2$, $r^2 \cdot r^2 = \pi\varrho r^2$. The structure of the free d-generated bicommutative algebra F_d and its most important numerical invariants were described by Dzhumadil'daev, Ismailov, and Tulenbaev [5]. Translating one of their results, if F_d is generated by $X_d = \{x_1, \dots, x_d\}$, then F_d^2 is isomorphic to the subalgebra of $K[Y_d, Z_d]$ with basis all monomials of positive degree both in Y_d and Z_d. Hence bicommutative algebras are very close to commutative associative algebras. The next two theorems are joint with Bekzat K. Zhakhayev [4].

Theorem 1. ([4]) *Finitely generated bicommutative algebras over any field are weakly noetherian, i.e., satisfy the ascending chain condition for two-sided ideals. The algebra* F_1 *has one-sided ideals which are not finitely generated and hence is not noetherian.*

Theorem 2. ([4]) *The variety* $\mathfrak{B}$ *of bicommutative algebras over a field of arbitrary characteristic satisfies the Specht property, i.e., its subvarieties have finite bases for their polynomial identities.*

When $\text{char}K = 0$ this is an immediate consequence of the results in [5] and Theorem 1. In positive characteristic the proof uses the Higman-Cohen method ([6], [2]) which is one of the main tools to prove the Specht property for groups and algebras in positive characteristic.

By [5] the codimension sequence of the variety $\mathfrak{B}$ is $c_1(\mathfrak{B}) = 1$, $c_n(\mathfrak{B}) = 2^n - 2$, $n = 2, 3, \ldots$. Hence $\exp \mathfrak{B} = \lim_{n\to\infty} \sqrt[n]{c_n(\mathfrak{B})} = 2$.

Theorem 3. ([3]) *Over a field of characteristic* 0 *the variety* $\mathfrak{B}$ *is minimal. If* $\mathfrak{V}$ *is a proper subvariety of* $\mathfrak{B}$ *satisfying a polynomial identity* $f = 0$ *of degree* k*, then* $\exp \mathfrak{V} = 1$ *and the codimension sequence* $c_n(\mathfrak{V})$*,* $n = 1, 2, \ldots$*, is bounded by a polynomial in* n *of degree* $k - 1$*.*

Theorem 4. ([3]) *Over a field of characteristic* 0 *the variety* $\mathfrak{B}$ *is generated by the two-dimensional algebra* $A_{1,-1}$*.*

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